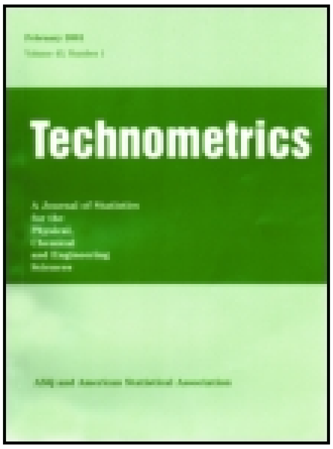


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# Self-starting monitoring scheme for Poisson count data with varying population sizes

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**Abstract**

In this paper we consider the problem of monitoring Poisson rates when the population sizes are time-varying and the nominal value of the process parameter is unavailable. Almost all previous control schemes for the detection of increases in the Poisson rate in Phase II are constructed based on assumed knowledge of the process parameters, e.g., the expectation of the count of a rare event when the process of interest is in control. In practice, however, this parameter is usually unknown and not able to be estimated with a sufficiently large number of reference samples. A self-starting EWMA control scheme based on a parametric bootstrap method is proposed. The success of the proposed method lies in the use of probability control limits, which are determined based on the observations during rather than before monitoring. Simulation studies show that our proposed scheme has good in-control and out-of-control performance under various situations. In particular, our proposed scheme is useful in rare event studies during the start-up stage of a monitoring process.

**Keywords:** Average run length; Healthcare surveillance; Poisson process; Probability control limits

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# 1 Introduction

In recent studies of statistical process control (SPC), monitoring the occurrence rate of a rare event from a sequence of counts has been one of the hot issues, e.g., the detection of changes in the rate of nonconformities in precise machining and manufacturing. In particular, in healthcare surveillance, interest is in detecting an increase in the mortality/incidence rates in primary care (Aylin et al. 2003), and the number of cancer patients (Krieger 2008; Han et al. 2010).

To detect changes in the occurrence rate of an adverse event, one usually assumes that the counts of events are (conditionally) independent Poisson random variables given the corresponding sample size. When the sample size is a constant, detecting a change in the rate is equivalent to detecting a change in the Poisson mean. Various control schemes developed for such a case can be found in Lucas (1985), Frisén and De Maré (1991) and White and Keats (1996). In healthcare surveillance, however, the sample sizes (i.e., the population sizes) are often time-varying. Increasing attention has thus been paid to the problem of monitoring the occurrence rate of an adverse event with time-varying sample sizes in prospective analysis (called Phase II). Related studies have been reported by Dong et al. (2008), Mei et al. (2010), Ryan and Woodall (2010), Shu et al. (2011), Zhou et al. (2012) and Shen et al. (2013). For a review of surveillance of nonhomogeneous Poisson processes, one may refer to Purdy et al. (2015).

All of these prior works have been primarily focused on the Phase II study with the assumption that the in-control (IC) value of the parameter, e.g., the occurrence rate, can be exactly known or accurately estimated based on a sufficient number of historical observations. Unfortunately, in many applications, there is no such information available and the monitoring of a Poisson process is to be activated at the start-up stage, i.e., the historical records for the estimation may be very limited. Recent literature reviews by Jensen et al. (2006) and Psarakis et al. (2014) provided



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thorough discussions of the effects of parameter estimation on control-chart performance. See also Testik (2007) and Castagliola and Wu (2012) for monitoring Poisson data when the parameters are estimated. They concluded that use of control charts with estimated parameters can produce a large bias in the IC average run length (ARL) when the number of reference samples is small and reduce the sensitivity of the chart in detecting process changes as measured by the out-of-control (OC) ARL. In many cases, it may not be feasible to wait for the accumulation of sufficiently large samples because the users want to monitor the process at the start-up stages. Therefore, a control scheme is required for monitoring the Poisson rate with limited historical information.



When it comes to the situation that sufficient information for parameter estimation is unavailable, self-starting methods are often used that update the parameter estimates along with new observations and simultaneously check for the OC conditions. Hawkins (1987) developed self-starting CUSUM charts for univariate normal data, and Quesenbery (1991a, 1995) discussed Shewhart equivalents. Later on, Quesenberry (1997), Schaffer (1998), Sullivan and Jones (2002), Hawkins and Maboudou-Tchao (2007) and Capizzi and Masarotto (2010) proposed self-starting charts for multivariate applications. Zou et al. (2007) further extended the self-starting technique to the application of profile monitoring. For monitoring of Poisson rates, control schemes were proposed by Quesenberry (1991b) and Hawkins and Olwell (1998). As we will show, Quesenberry's method is associated with an unsatisfactory IC ARL performance when the expectation of the Poisson counts is not sufficiently large. The method proposed by Hawkins and Olwell (1998) has satisfactory IC and OC ARL performance, but it is only applicable for the monitoring of Poisson rates when the sample sizes are constant over time. Its extension to the case with time-varying sample sizes is not straightforward, requiring careful consideration in the future. The main difficulty of designing an appropriate self-starting chart is that the control limit is difficult to determine without a pivotal test statistic when the process observations are Poisson distributed. In other words, in monitoring of a Poisson process, the IC distribution of charting statistics is usually not free of the underlying Pois-

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son rate and thus the control limit is difficult to determine in order to obtain a desired IC average run-length and a satisfactory OC average run-length.

To this end, we propose a self-starting exponentially weighted moving average (EWMA) control scheme for monitoring the occurrence rate of an adverse event when the IC Poisson process parameter is not known. In the proposed control scheme, it is not necessary to collect a large number of Phase I observations before the monitoring is activated (although it still requires one to collect a few preliminary samples). Similar to Shen et al. (2013), probability control limits, which are determined at each time point after observing the sample size, are used for the monitoring. Because we are considering the problem of unknown process parameters, the probability control limits can no longer be determined based on the methods provided by Shen et al. (2013), which require knowledge of the IC Poisson process. Therefore, a parametric bootstrap approach is developed for determining appropriate control limits. The proposed monitoring scheme has satisfactory average IC performance even when the reference sample is extremely small, which indicates good potential for its application. Although only the EWMA-type self-starting control chart is discussed in this paper, our ideas can be applied to any other competing control charts such as Shewhart charts and CUSUM charts.

The remainder of this paper is presented as follows. In Section 2, we first discuss the statistical assumptions and review some related work, then the model is formally defined and the proposed methodology is presented. Performance studies are reported in Section 3, followed by a demonstration of the proposed control scheme in a specific healthcare surveillance example. Finally, we conclude the paper with some discussion in Section 5.

# 2 Self-starting charts for monitoring Poisson rates

## 2.1 Model assumptions

Let $X_t$ be the count of an adverse event during the fixed time period $(t-1, t]$. For simplicity, we will call it the count of the event at time $t$. Suppose the $X_t$ values independently follow the Poisson distribution with the mean $\theta n_t$ conditional on $n_t$, where $\theta$ and $n_t$ denote the occurrence rate of the event and sample size at time $t$, respectively. To detect an abrupt change in the occurrence rate from $\theta_0$ to another unknown value $\theta_1 > \theta_0$ at some unknown time $\tau$, it is usually assumed that there are $m_0$ independent historical (reference) observations, $\{(X_i, n_i)\}_{i=-m_0+1}^{0}$, and the $i^{\text{th}}$ future observation, $X_i$, is collected over time following the change-point model

$$X_i \overset{\text{i.d.}}{\sim} \begin{cases} \text{Poisson}(\theta_0 n_i | n_i) & \text{for } i = -m_0+1, \ldots, 0, 1, 2, \ldots, \tau-1, \\ \text{Poisson}(\theta_1 n_i | n_i) & \text{for } i = \tau, \ldots, \end{cases} \tag{1}$$

where the symbol $\overset{\text{i.d.}}{\sim}$ means "independently distributed". The objective is to detect the change as soon as possible after it occurs by using the sequentially observed counts. In this study, we are interested in the departures from stability taking the form of upward sustained shifts in $\theta$ when assuming that $n_t$ is a known value at each time point $t$ while $\theta_0$ and $\theta_1$ are not known exactly a *priori*.

## 2.2 Existing work

We review four "off-the-shelf" procedures for the monitoring of the Poisson rates in this section. Recall that in Phase II analysis the process parameter (e.g., the Poisson rate) is usually assumed to be known or well estimated based on a sufficiently large number of reference samples. A tra-

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ditional way, termed the "TR-1" method in the following, is to estimate $\theta_0$ with the $m_0$ historical observations, i.e., $\hat{\theta}_0 = \sum_{j=-m_0+1}^{0} X_j / \sum_{j=-m_0+1}^{0} n_j$. Then, we regard $\hat{\theta}_0$ as the "true" parameter and set up the monitoring scheme. When the historical data are very limited, i.e., $m_0$ is small, the estimate may significantly deviate from the true value. As a result, such a monitoring scheme may have unsatisfactory performance.

Another benchmark is Shen et al.'s (2013) procedure (abbreviated as TR-2), in which probability control limits are considered. Those control limits are determined one-by-one during monitoring. However, the value of $\theta_0$ needs to be specified so that the probability control limits $h_t$ ($t = 1, 2, \ldots$) can be determined through a simulation approach by generating the Poisson variables with the rate $\theta_0$. Without knowledge of $\theta_0$, we can only arbitrarily assume a value for $\theta_0$ (or one estimated by using a few historical sample) and determine the probability control limits based on the "true value" in such a monitoring scheme. As a result, the performance of this scheme depends heavily on how close the estimated in-control value is to the value of $\theta_0$.

In regard to the limited historical information, a normal approximation method was also suggested by Quesenberry (1991b) for monitoring the Poisson rate. We call it the "TR-3" method hereafter. In this method, observed Poisson counts are transformed to approximately standard normally distributed variables. Specifically, the sequential observations $(X_1, n_1)$, $(X_2, n_2)$, $\ldots$ are transformed to $Q$ statistic by the following formulas,

$$\begin{cases} u_t = B(X_t; s_t, n_t/N_t) \\ Q_t = \Phi^{-1}(u_t), \ \text{for } t = 1, 2, \ldots, \end{cases}$$



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where $s_t = \sum_{k=1}^{t} X_k$, $N_t = \sum_{k=1}^{t} n_t$,

$$B(x; s, p) = \begin{cases} 0, & x < 0 \\ 1, & x \geq s \\ \sum_{k=0}^{[x]} b(k; s, p), & 0 \leq x < n \end{cases}$$

and

$$b(x; s, p) = \begin{cases} \begin{pmatrix} s \\ x \end{pmatrix} p^x (1-p)^{s-x}, & x = 0, 1, \ldots, n, \\ 0, & \text{otherwise.} \end{cases}$$

These variables $Q_1, Q_2, \ldots$ are independent and approximately normally distributed statistics given the value of $\hat{\theta}_0$ and hence an EWMA monitoring scheme for standardized normal variables can be applied directly. However, we show that the accuracy of the method depends heavily on the values of $\theta_0 n_t$ and hence the ARL performance of the corresponding monitoring scheme cannot be guaranteed, especially when the products $\theta_0 n_t$ are not sufficiently large.

In addition, another self-starting control scheme, termed the "TR-4" method, was proposed for Poisson counts by Hawkins and Olwell (1998). In this method, Poisson counts are assumed to follow the same distribution, i.e., $X_t \sim \text{Poisson}(u_0)$ for $t = 1, 2, \ldots$, while the in-control mean $u_0$ is unknown. If the process is in control, then

$$X_t \sim \text{Binomial}(W_t, 1/t),$$

where $t$ is the sample size and $W_t = \sum_{i=1}^{t} X_i$. A cumulative probability can then be obtained as follows. Let $A_t = \Pr[B_t \leq X_t]$ where $B_t \sim \text{Binomial}(W_t, 1/t)$. In order to convert $A_t$ to an equivalent standardized form, one estimates $u_0$ using the average of the preliminary observations. Suppose the estimator is $\hat{u}_0$. Then $X_t$ is transformed to a Poisson variate $Y_t$ with parameter $\hat{u}_0$ such that

$$\min_{Y_t} \left| \sum_{j=0}^{Y_t} \frac{e^{-\hat{u}_0} \hat{u}_0^j}{j!} - A_t \right|.$$

We let $Y_t = X_t$ when $A_t = 1$. The Poisson variate $Y_t$ is then monitored by the EWMA scheme. Our simulation studies demonstrate that under the scenario of constant sample sizes, this scheme has competitive IC and OC performances. However, its application is restricted to monitoring the Poisson rates with constant sample size. When the population size is time-varying, it is clear that the arguments above do not hold. Our simulation (not reported here but available from authors) show that some "naive" modifications cannot yield satisfactory IC performances under the scenario of time-varying sample sizes.



## 2.3 Proposed method

Recall that the monitoring of the Poisson count data is activated at time $t = 1$ with $m_0$ historical counts of events. That is, at time $t = 0$, there have been $m_0$ paired observations recorded as $\{(X_{-m_0+1}, n_{-m_0+1}), \ldots, (X_0, n_0)\}$. We let $\hat{\theta}_t$ represent the estimated occurrence rate at time $t$ based on the previous observations, i.e.,

$$\hat{\theta}_t = \frac{\sum_{j=-m_0+1}^{t-1} X_j}{\sum_{j=-m_0+1}^{t-1} n_j}, \quad t = 1, 2, \ldots. \tag{2}$$

Alternative estimates of $\theta_t$ can also be used. For example, $\hat{\theta}_t$ can be a weighted average of the previous observations, assigning larger weights to later observations. Further we define the standardized counts $S_t = (X_t - n_t\hat{\theta}_t)/\sqrt{n_t\hat{\theta}_t}$ and construct the EWMA-type charting statistics as

$$Z_t = \max\{0, (1-\lambda)Z_{t-1} + \lambda S_t\}, \quad t = 1, 2, \ldots, \tag{3}$$

where $\lambda \in (0, 1]$ is the smoothing parameter and $Z_0 = 0$. Here, we use a reflecting barrier, as did Ryan and Woodall (2010), to avoid inertia problems (Woodall and Mahmoud 2005). Clearly, when the process occurs some change at $\tau$, a large value of $Z_t$ would be used to trigger an alarm.

To construct a control chart, it is critical to determine the control limits so that a charting scheme has satisfactory run-length behavior. As indicated by Hawkins and Maboudou-Tchao (2007) and Qiu (2014; Chapter 6), it can be insufficient to summarize run-length behavior by the ARL, especially for self-starting control charts. The IC run length distribution is usually considered to be satisfactory if it is close to the geometric distribution. A chart is usually unacceptable if the specified IC ARL is obtained with an elevated probability of false alarms with short run lengths as compared to a geometric distribution. An excessive number of false alarms would hurt an operator's confidence in valid alarms. To this end, we want to find the control limits so that the conditional probability that the charting statistic exceeds the control limit at present, given that there is no false alarm before the current time point, is a pre-specified constant. Let $h_t$ be a sequence of control limits. Theoretically, the control limits $h_t$'s should satisfy the following equations,

$$\begin{aligned} &\Pr(Z_1 > h_1(\alpha) \mid n_1) = \alpha, \\ &\Pr(Z_t > h_t(\alpha) \mid Z_i < h_i(\alpha),\ 1 \le i < t,\ n_t) = \alpha \ \text{ for } \ t > 1, \end{aligned} \tag{4}$$

where $\alpha$ is the pre-specified conditional false alarm rate.

This is somewhat analogous to performing a hypothesis test with the type-I error $\alpha$ at each time point $t$. The approach of determining control limits to maintain a constant conditional false alarm rate was originally proposed by Margavio et al. (1995) and has been used by Hawkins et al. (2003). Zou and Tsung (2010) gave a related discussion. Traditionally, the dynamic control limits can be obtained by using a Markov chain approximation or Monte Carlo simulation techniques because the charting statistics considered in the literature are usually pivotal quantities.



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However, as we have indicated in Section 1, it is not easy to approximate $h_t(\alpha)$ because the IC distributions of $Z_t$'s (given $n_t$) depend on the parameter $\theta_0$. To this end, we propose to use a parametric bootstrap iteratively. That is, at each time point $t$, we generate pseudo Poisson observations with the expectation $n_t\hat{\theta}_t$ and then find $h_t$ by approximately controlling the probabilities in (4).

To clearly explain the proposed scheme, we first consider the monitoring at the time point $t = 1$. According to the observed $n_1$ and the estimated occurrence rate $\hat{\theta}_1$, where $\hat{\theta}_1 = \sum_{j=-m_0+1}^{0} X_j / \sum_{j=-m_0+1}^{0} n_j$, we are able to randomly generate pseudo observations $Y_{i,1}$, where $i = 1, 2, \ldots, N$ and $N$ is a sufficiently large integer, from Poisson($n_1\hat{\theta}_1$) and have the corresponding $S_{i,1} = (Y_{i,1} - n_1\hat{\theta}_1)/\sqrt{n_1\hat{\theta}_1}$. Then we can obtain a sample of pseudo charting statistics $\mathbf{R}(1) = \{R_1(1), R_2(1), \ldots, R_N(1)\}$, where $R_i(1) = \lambda S_{i,1}$. We let the elements in $R(1)$ be ranked in ascending order to rewrite the vector $\mathbf{R}(1)$ as $(R_{[1]}(1), \ldots, R_{[N]}(1))$. Then we can use $R_{[H]}(1)$ to approximate $h_1(\alpha)$, where $H = \lfloor N(1-\alpha) \rfloor$ and $\lfloor \ \rfloor$ is the rounding symbol.

If $Z_1 > h_1$, $X_1$ is declared to be OC and an alarm is triggered accordingly. Otherwise, we continue the monitoring to the next time point $t = 2$ and estimate $\theta_2$ based on $\hat{\theta}_1$ and $X_1$, i.e., $\hat{\theta}_2 = \left(\hat{\theta}_1 \sum_{j=-m_0+1}^{0} n_j + X_1\right) / \sum_{j=-m_0+1}^{1} n_j$. At time $t = 2$, to determine the control limit $h_2$, we first should restrict the feasible values of $R_i(1)$ to satisfy (4). More specifically, we keep a part of $\mathbf{R}(1)$, $(R_{[1]}(1), \ldots, R_{[H]}(1))$, as the space of feasible values of $R_{[i]}(1)$ and randomly bootstrap $N$ variables of $R_i(1)$ from the space to make up an updated $N-$dimensional vector $\mathbf{R}(1)$. The control limit $h_2$ can be similarly determined by repeating the process of random number generation of Poisson($n_2\hat{\theta}_2$) random variables.

The proposed procedure is summarized as follows.

1. If $Z_{t-1} \leq h_{t-1}$, estimate $\hat{\theta}_t$ based on the past observations $\{(X_1, n_1), \ldots, (X_{t-1}, n_{t-1})\}$.

2. Randomly generate $Y_{i,t} \sim$ Poisson($n_t\hat{\theta}_t$), $i = 1, \cdots, N$, and obtain $\mathbf{R}(t) = \{R_1(t), \ldots, R_N(t)\}$,

where $R_i(t) = (1-\lambda)R_i(t-1) + \lambda S_{i,t}$ and $S_{i,t} = (Y_{i,t} - n_t\hat{\theta}_t)/\sqrt{n_t\hat{\theta}_t}$. After sorting the elements of $\mathbf{R}(t)$ in ascending order, the control limit can be determined to be the value of $R_{[H]}(t)$.

3. Compare the charting statistic $Z_t$ with $h_t$ and decide whether the monitoring should be continued or not. If continuing to the next time point, update the $\mathbf{R}(t)$ by randomly selecting $R_i(t)$'s from $(R_{[1]}(t-1), \ldots, R_{[H]}(t-1))$ and go back to step 1 to estimate $\hat{\theta}_{t+1}$.

The procedure described above can be considered as a parametric bootstrap method, with a Poisson model as the parametric assumption and a resampling technique for approximating the sampling distribution of the charting statistics. It should be emphasized here that our procedure significantly differs from the approach of Margavio et al. (1995) and others in the sense that the control limits in our procedure are determined on-line along with the process observations rather than decided upon before monitoring. That is, our control limits are data-driven due to the use of $\hat{\theta}_t$ and $n_t$ at the time point $t$. Due to the discreteness of the Poisson distribution, the conditional false alarm probabilities will not generally be equal to $\alpha$, but we will show that the approximations are always close.

Although the proposed chart is a self-starting scheme in the sense that it can be implemented at the start-up of a process, we believe that starting testing with an $m_0$ that is too small is not a good idea. A small $m_0$ value would result in a severe "masking-effect" if a short-run change occurs (Hawkins et al. 2003). Rather, we suggest that a practitioner should gather a modest number of observations through a Phase-I study and prior knowledge to obtain at least an initial verification that the process is actually stable, and only then should the practitioner start formal charts. Jones-Farmer et al. (2014) provided a discussion of the issues and benefits related to Phase I studies. Our empirical results show that obtaining a satisfactory monitoring performance with our method may require at least 20-40 IC observations (of course the more the better) before the change actually occurs; this number also depends on $\theta_0 n_t$ and shift magnitudes.



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In comparison with other existing work, our proposed method requires a considerable amount of computing time because our control limits are obtained on-line. Today's computing power has improved dramatically and it is computationally feasible to implement our proposed chart. The computer codes implementing the proposed scheme in Fortran are available upon request.

# 3 Performance study

In order to investigate the performance of the proposed EWMA-type self-starting control scheme and that of the traditional schemes with the limits determined based on some assumed values of the process parameters, we report some simulation results in this section to show that (1) the TR-1, TR-2, and TR-3 methods are not suitable because their charting statistics are not pivotal quantities and hence their IC performance is not acceptable; (2) our proposed control scheme has satisfactory performance and, when the sample sizes are constant, its performance is competitive to that of the TR-4 method under the scenario of constant sample sizes; (3) The proposed method is effective for the case with with varying population sizes.

## 3.1 IC performance

We first investigate the IC performance of our proposed scheme and that of the traditional control schemes. Note that the factors related to the IC performance of the monitoring schemes are (1) the number of historical observations $m_0$; (2) the smoothing parameter $\lambda$; (3) the sample size at each time point $n_t$; and (4) the IC occurrence rate $\theta_0$. Since $n_t$ is known at time $t$, whether its value is time-varying or constant does not impact the performance of our proposed control scheme. For simplicity, we first assume that $n_t$ is a constant value in the simulations. The performance of the



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control schemes with time-varying sample sizes will be presented at the end of this section.

In the first simulation study, we assess the influence of the amount of historical data $m_0$ and the smoothing parameter $\lambda$ on the performance of the proposed control scheme under a specific conditional false alarm rate $\alpha$ with a known constant value of $n_t$. Both the historical data and the Poisson counts "observed" in the monitoring process are generated from the same Poisson distribution with a specific value of $\theta_0$. In each simulation trial, we determine the control limits with the specific value of $\hat{\theta}_0$, which is estimated based on the preliminary observations, and then obtain one value of run length. Note that we do not obtain multiple values of run lengths based on the same set of preliminary observations in this section or in Sections 3.2 and 3.4. The obtained IC ARL value is actually a random variable depending on the preliminary observations. Hence, we estimate the average IC ARL, termed "$\text{AARL}_0$", by repeating $M$ simulation trials (in each trial we generate different preliminary observations). Performance of interest is then evaluated by this $\text{AARL}_0$. In our approach and for TR-2, $N$ should be a sufficiently large value in order to determine the appropriate probability control limits. We set $\theta_0 = 1$, $n_t = 20$, $\alpha \in \{0.005, 0.002, 0.001\}$, $N = 20,000$ and $M = 2,000$. For $m_0 \in \{10, 20, 50, 100\}$ and $\lambda \in \{0.05, 0.10, 0.50, 0.80\}$, we have the estimated average IC ARLs ($\text{AARL}_0$) recorded in Table 2, as well as the absolute relative error, $\delta = |\text{AARL}_0 - \text{ARL}_0|/\text{ARL}_0$, where $\text{ARL}_0$ is the nominal ARL.

The results presented in Table 2 indicate that the values of $\text{AARL}_0$ associated with the TR-1 and TR-2 methods depend heavily on both the number of historical observations $m_0$ and the smoothing parameter $\lambda$. In particular, a larger $\lambda$ value yields better average IC ARL performance for the two methods. For each value of $\lambda$, the performance of TR-1 and TR-2 can be improved significantly by increasing the number of historical observations, which is consistent with our intuition. When $\lambda$ is large (e.g., $\lambda = 0.8$), $m_0 = 100$ is sufficient for the estimation of $\theta_0$ in TR-2. However, for the TR-1 method, the necessary amount of historical data increases along with the desired value of $\text{ARL}_0$. For example, with $\lambda = 0.8$, $m_0 = 50$ is sufficient for the estimation of $\theta_0$ when the

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desired $\text{ARL}_0$ is 200 (i.e., $\alpha$ is equal to 0.005). However, $m_0 = 50$ is far from enough data when the desired $\text{ARL}_0$ is 500 or 1000. As a result, we conclude that the TR-1 and TR-2 methods are not suitable for monitoring activated at the start-up stage when the process parameter is unknown. The performance of the TR-3 method is rarely impacted by $m_0$ but depends significantly on $\lambda$. Similarly, a larger $\lambda$ value leads to better average IC ARL performance for the TR-3 method. However, for the TR-3 method, its average IC ARLs are far from the desired value even with a large value of $\lambda$ when $\theta_0 n_t = 20$. We hence conclude that the performance of the TR-3 method is largely affected by the values of $\theta_0 n_t$, which will be further explored later in this section.

In contrast, the TR-4 method and our proposed control scheme have remarkable IC performance. That is, the values of the average IC ARLs estimated for these two methods are close to the desired value $\text{ARL}_0$ for all listed $m_0$ and $\lambda$. Even when the number of available observations is very limited, e.g., $m_0 = 10$ in our simulation, the TR-4 method and the proposed control scheme can provide us with satisfactory performance. Though the estimated $\text{AARL}_0$ is a little bit different from its desired value, its absolute relative error ($\delta$) is always smaller than 0.1. The two methods are thus considered to have potential in applications when large reference samples are not available.

Next we analyze the impact of $\theta_0$ and $n_t$ on the performance of the proposed control scheme when fixing $m_0 = 20$ and $\lambda = 0.1$. We note that the performance of both the proposed control scheme and the TR-2 method is actually not related to the sample size $n_t$; see Shen et al. (2013) for a discussion. The most significant difference between the two schemes is that the true $\theta_0$ value has to be known in the TR-2 method while it is not required in the proposed one. Accordingly, the difference between the performance of the two schemes relies on how the estimated value of $\theta_0$ is going to affect the probability control limits and further impact the performance of the TR-2 method. In the following simulation study, we set $\theta_0 \in \{0.1, 1, 5, 10\}$ and $n_t \in \{20, 200, 2000\}$. Other factors have the same values as the previous simulation study. The simulation results are presented in Table 3.



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Let us first focus on the performance of the TR-1 and the TR-2 methods. In Table 3, we do not observe any significant patterns between the values of the estimated $\text{AARL}_0$ values and the values of $(\theta_0, n_t)$. Hence we conclude that the values of $\theta_0$ and $n_t$ do not play important roles in the performance of the two methods. Next, we consider the TR-3 method. Its performance is significantly improved by increasing the value of $\theta_0$ or that of $n_t$, as we have mentioned before. When the product $\theta_0 n_t$ is larger than or equal to $10,000$, the average IC ARLs resulting from the TR-3 method are very close to the desired values (see the 5th, the 8th, and the 11th columns under $\theta_0 = 5$ and $\theta_0 = 10$ in Table 3) even when $m_0 = 20$ and $\lambda = 0.1$. As a result, we can conclude that the control limits of the TR-3 method depend heavily on the values of $\theta_0 n_t$. Satisfactory performance may be expected for TR-3 when the count of occurrence (represented by $\theta_0 n_t$) is sufficiently large. This is not surprising, but its performance becomes significantly worse when the product $\theta_0 n_t$ is small. In other words, the TR-3 method should not be applied in the case of rare events. Note that, for a Binomial distribution, its normal approximation should work quite well when the binomial mean (i.e., $\theta_0 n_t$) is larger than 10. However, according to our simulation study, the TR-3 method is effective only when the count of occurrence is larger than 2000. We then explain the failure of the TR-3 method as follows when the count of occurrence is not sufficiently large. Recall that in the TR-3 method we make the transformation $Q = \Phi^{-1}(u_t)$. The resulting $Q$ is normally distributed with a standard deviation close to 1 but an offset non-zero mean. The systematically wrong IC mean leads to a quick OC signal in the cumulative charts, e.g., the EWMA chart. The problem here lies on the "≤" in the binomial cumulative distribution function. Therefore, if we want to resurrect the TR-3 method in our problem, applying a continuity correction with the TR-3 method may be an effective way.

Now, turning our attention to the results corresponding to the TR-4 method and the proposed control scheme, the simulated results indicate that the two methods both have stable performance under different values of $\theta_0$ and $n_t$. In particular, their performance is very impressive when the expected count of occurrences is small, e.g., $\theta_0 n_t = 2$. Again, with the two methods, the absolute percentage error of the estimated average IC ARLs from the desired value is smaller than 10%.

Taking into account their IC performance in the previous simulation study, we can provide a comprehensive evaluation of the TR-4 method and the proposed self-starting control scheme as follows. First of all, the two methods have comparable IC performance when sample sizes are constant. They are able to bring us robust and satisfactory IC run length performance under various situations. Because of its robustness, the TR-4 method and the proposed scheme are much better than (or competitive to) the other methods when there is not a sufficient number of reference samples. Their advantage is even more prominent in the monitoring of rare events as shown by the simulation results.

Thus far, all the presented simulations are based on a constant $n_t$ for simplicity. In the following, we further show that the proposed control scheme also has good performance when sample sizes are time-varying. Referring to the previous studies, the time-varying sample sizes are depicted by three different scenarios based on two logistic models suggested by Mei et al. (2011) and one uniform distribution model previously used by Ryan and Woodall (2010). Following the work by Mei et al. (2011), we let $c_1 = 13.8065$, $c_2 = 11.8532$, $c_3 = 26.4037$ and define the three scenarios as



**Scenario (A)** The logistic model, $n_t = (20c_1)/\{1+\exp[-(t-c_2)/c_3]\}$, is used to describe a scenario of increasing time-varying sample sizes.

**Scenario (B)** The logistic model, $n_t = (c_1/2.4)/\{1 + \exp[(t - c_2)/c_3]\} + 18$, is used to describe a scenario of decreasing time-varying sample sizes.

**Scenario (C)** The time-varying sample sizes are uniformly distributed, i.e., $n_t \sim [15, 20]$.

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For each scenario, we set $\lambda = 0.1$, $m_0 = 20$, $\alpha = 0.005$ and $\theta_0 = (0.1, 1, 5, 10)$. The performance is presented in Table 4, which verifies the capability of the proposed control scheme to control the average IC ARL when the sample sizes are time-varying.

## 3.2 OC performance

In this section, we study the OC performance of the proposed control scheme. For comparison, we also apply the TR-4 method. The other three"off-the-shelf" procedures are not investigated because of their unacceptable IC performance. We set $m_0 = 20$ and assume $\theta_0$, $n_t$ to be 1 and a constant value of 20, respectively.

We investigated the OC performance under different OC conditions, i.e., the rate of event occurrence changes from $\theta_0$ to $\theta_1$ at time $\tau$. By assuming $\tau = 21, 41$, $\lambda = 0.1, 0.5$, and changing the value of $\theta_1$ from a low of 1.025 to a maximum of 1.5, we obtained the simulation results in Table 5. The results indicate that, though the OC performance of our proposed control scheme is slightly better than that of the TR-4 method, the two schemes in general have similar OC performance. We thus conclude that, when sample sizes are constant over time, our proposed control scheme has competitive IC and OC performance with the TR-4 method.

## 3.3 Conditional IC ARL

The investigated average IC ARLs in our simulation study are averaged over all possible sets of historical observations. More specifically, for each set of historical data, one value of run length was obtained and the estimated average IC ARL is an average of these run lengths. Therefore, the average IC ARL we considered is actually the in-control expected ARL.



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However, as indicated by Saleh et al. (2015), Keefe et al. (2015), and others, the different historical data sets practitioners use result in varying estimates, control limits, and their corresponding IC ARLs. Hence the reliance on the expected IC ARL averages across the practitioner-to-practitioner variability. Therefore, the variability of the IC ARLs corresponding to different historical data sets should be kept at a low level in order to have a satisfactory in-control performance.

To this end, in the following, we further conducted a simulation study to show that our proposed control scheme is able to ensure low levels of variation in the IC ARL values among different historical data sets. We set $n_t = 20$, $\theta_0 = \{0.1, 1\}$, $\lambda = 0.1$, $\alpha = \{0.005, 0.002\}$, and $m_0 = \{10, 20, 30, 50\}$. Under each setting, we generated 100 sets of historical observations. For each set, we simulated 400 charts and estimated the corresponding IC ARL. The average IC ARL (termed AARL) and the standard deviation of IC ARL (termed SDARL) under each setting were estimated based on the 100 IC ARLs. As Saleh et al. (2015) discussed, having SDARL values of a chart within 10% of its intended in-control ARL value is reasonably good performance, although it still represents significant variation.

Table 6, showing the AARL and SDARL under different settings, indicates that our proposed control scheme is able to guarantee relatively small variation in the IC ARL values among different historical data sets, even when the number of historical observations is very limited and the Poisson mean is very small (e.g., $m_0 = 10$ and $\theta_0 = 0.1$). The reason is as follows. In our control scheme, given a set of historical observations, we use an online parametric bootstrap to update the estimate of $\theta_0$ based on the new observation at each time point. Hence, the control limit for the next time point is determined based on all the available observations. The IC ARL corresponding to each given set of historical data is hence close to its desired value under each setting, and accordingly a relatively small value of SDARL is obtained with our scheme compared to estimating the Poisson parameters on only the preliminary data.

### 3.4 Effect of choice of $N$ on the proposed method

We have indicated that $N$ should be sufficiently large in our proposed method to obtain accurate probability control limits, and hence we set $N = 20,000$ in our simulation studies. But how would the value of $N$ affect the performance of our proposed control scheme? We investigated this question through a simulation study. The parameters $\theta_0$, $n_t$, and $\lambda$ may affect the performance of our proposed method under each value of $N$. Hence, we fix $m_0 = 20$ and $\alpha = 0.005$ ($E(\text{ARL}_0) = 200$). For $\theta_0 \in \{1, 5\}$, $n_t \in \{20, 200\}$, $\lambda \in \{0.10, 0.40, 0.80\}$, and $N = \{200, 500, 1000, 2000, 5000, 10000\}$, we have estimated $\text{AARL}_0$'s as well as the absolute relative error from $E(\text{ARL}_0)$ recorded in Table 7.

Table 7 indicates that, to obtain satisfactory probability control limits, the required value of $N$ increases with the value of $\lambda$, but has little relationship with the values of $\theta_0$ and $n_t$. Specifically, $N$ should be no smaller than 2000 when $\lambda = 0.1$, but must be larger than 5000 when $\lambda = 0.80$. As a result, in applications, a larger $N$ should be used when $\lambda$ approaches 1. According to our simulation study, $N = 10000$ is large enough for $\lambda = 0.80$ but may need to be 20000 if $\lambda = 1$.. For a smaller $\lambda$, e.g., $\lambda \leq 0.4$ in Table 7, $N \geq 5000$ is sufficiently high for our approach. The value of $N$ can be quite large without causing computational difficulty.

## 4 An application

In this section, the example of melanoma, one of the skin cancers, is used to demonstrate the application of the proposed EWMA-type self-starting control scheme. Skin cancer is the most common type of cancer and accounts for about half of all cancers in the United States[1]. According

[1] http://www.cancer.org/Cancer/CancerCauses/SunandUVExposure/skin-cancer-facts

to the National Cancer Institute, more than 2,000,000 Americans develop skin cancer annually [2]. There are three most common types of skin cancer. Among these, melanomas of the skin are considered the most dangerous because they are more likely to spread to other parts of the body. Over all ages, melanoma is the eighth most common type of cancer among men as well as among women. In the state of New York, from 2005 through 2009, approximately 1,957 men and 1,493 women were diagnosed with melanoma each year, accounting for approximately 4 percent of cancers among men and 3 percent of cancers among women. About 9,784 men and 7,467 women who resided in New York had a diagnosis of melanoma within the past five years.

The New York State Department of Health (NYSDOH) collected the data related to the melanoma each year, including the number of incidences and the population size in each county in New York State, from 1976 to 2010 and further reported the annual information through its official website[3]. The documented dataset is available in the supplementary material. Figure 1 presents for females the time series plots of (a) the counts of incidence, (b) the population sizes (in the units of 100,000) and (c) the incidence rates of melanoma per 100,000 population in Manhattan during the past 35 years. The population sizes experienced large changes over time. The incidence rate showed a significant growth trend starting from 1995 and the values remained high until the end of the records. The American Cancer Society (2012) believes that the increased rate could be related to the danger of indoor tanning, which has become more common in recent years.

According to the observed information, we chose to use model (1) and start the monitoring at the year 1986. The 19 observations during 1976-1994 are used as the historical data, i.e, we used $m_0 = 19$. Note that the validity of our method relies on the assumption that the observations are (at least approximately) Poisson distributed with time-varying sample sizes. The Poisson distribution is widely used for modelling the mortality/incidence counts in healthcare studies by, e.g., Schwartz

[2]http://www.cancer.gov/cancertopics/types/skin
[3]http://www.health.ny.gov/statistics/cancer/registry/table2/ b2melanomanewyork.htm

(1993), Schwartz et al. (1996), Ösby (2000), Brouhns et al. (2005), and Nakaya et al. (2005). However, it is quite difficult to statistically test for a heterogeneous Poisson process with rare observations, which is exactly the case in this example. Such a study is beyond the scope of this paper, but could be subject of future research. We chose $\lambda = 0.1$ and $\alpha = 0.01$. In addition to our proposed control scheme, all the traditional methods except the TR-4 method are applied in this example. The TR-4 method is not used here due to its limitation to monitoring Poisson rates with constant sample sizes. Table 1 presents the control limit (CL), the charting statistic (CS) corresponding to each year, and also the signalling status (S) (specifically, it refers to IC or OC) of the four methods. The proposed control scheme and the TR-2 method issued signals at the year 1996, whereas the TR-1 and TR-3 methods issued signals at the year 1997. The result is consistent with our observation, that is, a significant increase occurred after the year 1995. Through this example, we have illustrated the use of the proposed self-starting control scheme when only limited historical information is available.



## 5 Conclusion

With an increasing interest in the monitoring of Poisson rates, many studies have been conducted in Phase II, where the in-control process parameter is assumed to be known exactly. We consider the problem of monitoring the Poisson rates with time-varying sample sizes when the in-control process parameter is not available, which is the common situation in applications. In cases of an insufficient number of reference samples, the self-starting method is commonly used. However, there is an obstacle in applying the self-starting method in monitoring the Poisson rates with time-varying sample sizes, i.e., it is difficult to determine the control limit appropriately for the Poisson rates since the limit depends heavily on the unknown process parameter. Inappropriate assumptions or estimation of the parameter may lead to unacceptable average IC run length performance of

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the control charts, and/or reduce the sensitivity of the chart in detecting process changes. To overcome this hurdle, we propose a self-starting monitoring scheme based on the EWMA chart for monitoring the Poisson rate when the in-control parameter is unknown. The probability control limits discussed by Shen et al. (2013) are used, but determined with the simulation approach newly developed in our paper. The presented self-starting scheme can be readily applied to other traditional control charts, e.g., the CUSUM chart, through simple modification. The simulation studies conducted in our paper have verified the robust and satisfactory IC and OC run length performance under different settings. In particular, the proposed control scheme has a very good performance in the rare event situation when the monitoring is activated near the start-up stage (i.e., the number of reference samples is small).

Future research may be considered in the following directions. First of all, recall that in this paper we applied a simple EWMA-type charting statistic, as well as a naive estimate of the parameter at each time point. Obviously, more effective charting statistics, e.g., the weighted-likelihood-based EWMA statistic proposed by Zhou et al. (2012), can be integrated into our proposed scheme. Similarly, the estimation of the parameter may be improved, e.g., by using a moving average, in order to derive better performance of the control scheme. In addition, more work is needed to extend our method to Phase I analysis, in which detection of outliers or change-points in a set of historical observations and estimation of the baseline occurrence rate is of particular interest. Note that the performance of all control charts is impacted by the number of reference samples. As a result, it would be interesting to investigate the necessary amount of Phase I data to ensure the specified in-control performance in Phase II. Moreover, our bootstrap-based approach may be applied to design a GLRT-based control chart (see Naus and Wallenstein (2006) for reference) developed in the literature on temporal scan statistics.



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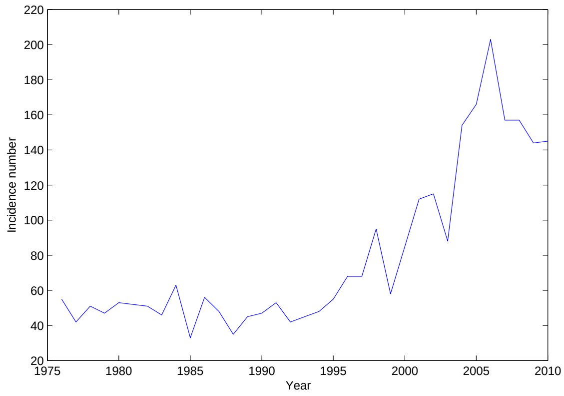
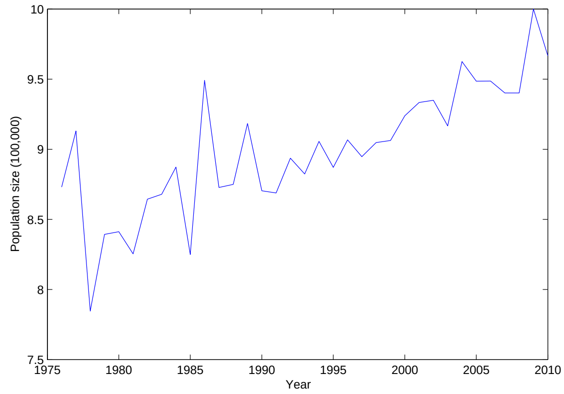
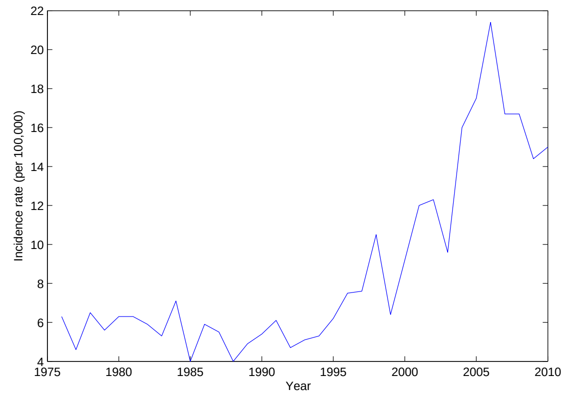



(a) Number of incidences




(b) Population size




(c) Incidence rate

Figure 1: Female melanoma incidence data. (a) Number of female incidences (b) Female population (c) Incidence rate.

Table 1: Estimated $AARL_0$ (with $\delta$ in parentheses) of the proposed control scheme, the TR-1 method, the TR-2 method, the TR-3 method, and the TR-4 method when $\theta_0 = 1$ and $n_t = 20$.

| | $\alpha$ [$ARL_0$] | 0.005 [200] | | | | 0.002 [500] | | | | 0.001 [1000] | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m_0$ | $\lambda$ | 0.05 | 0.10 | 0.50 | 0.80 | 0.05 | 0.10 | 0.50 | 0.80 | 0.05 | 0.10 | 0.50 | 0.80 |
| 10 | Proposed | 192 (0.04) | 184 (0.08) | 188 (0.06) | 190 (0.05) | 472 (0.06) | 464 (0.07) | 462 (0.08) | 467 (0.07) | 951 (0.05) | 965 (0.03) | 940 (0.06) | 941 (0.06) |
| | TR-1 | 831 (3.16) | 577 (1.89) | 507 (1.54) | 314 (0.57) | 2144 (3.29) | 1944 (2.89) | 1052 (1.10) | 1035 (1.07) | 4387 (3.39) | 3325 (2.33) | 2456 (1.46) | 1674 (0.67) |
| | TR-2 | 656 (2.28) | 583 (1.92) | 421 (1.11) | 346 (0.73) | 1859 (2.72) | 1641 (2.28) | 1129 (1.26) | 901 (0.80) | 3608 (2.61) | 3299 (2.23) | 2125 (1.13) | 1814 (0.81) |
| | TR-3 | 79.6 (0.60) | 88.8 (0.56) | 124 (0.38) | 144 (0.28) | 155 (0.69) | 183 (0.63) | 296 (0.41) | 348 (0.30) | 253 (0.75) | 320 (0.68) | 564 (0.44) | 663 (0.34) |
| | TR-4 | 197 (0.02) | 203 (0.02) | 192 (0.04) | 194 (0.03) | 496 (0.01) | 483 (0.03) | 531 (0.06) | 526 (0.05) | 1057 (0.06) | 901 (0.10) | 1029 (0.03) | 913 (0.09) |
| 20 | Proposed | 189 (0.05) | 190 (0.05) | 187 (0.06) | 188 (0.06) | 483 (0.04) | 470 (0.06) | 460 (0.08) | 490 (0.02) | 947 (0.05) | 955 (0.04) | 948 (0.05) | 932 (0.07) |
| | TR-1 | 664 (2.32) | 497 (1.49) | 331 (0.66) | 295 (0.48) | 1651 (2.30) | 1356 (1.71) | 827 (0.65) | 687 (0.37) | 3598 (2.60) | 2863 (1.86) | 1854 (0.85) | 1621 (0.62) |
| | TR-2 | 582 (1.91) | 524 (1.62) | 343 (0.72) | 306 (0.53) | 1542 (2.08) | 1437 (1.87) | 791 (0.58) | 636 (0.27) | 3346 (2.35) | 2854 (1.85) | 1539 (0.54) | 1368 (0.37) |
| | TR-3 | 77.0 (0.62) | 88.4 (0.56) | 128 (0.36) | 143 (0.29) | 151 (0.70) | 179 (0.64) | 293 (0.41) | 338 (0.32) | 247 (0.75) | 325 (0.68) | 554 (0.45) | 644 (0.36) |
| | TR-4 | 195 (0.03) | 189 (0.06) | 188 (0.06) | 198 (0.01) | 535 (0.07) | 501 (0.00) | 491 (0.02) | 517 (0.03) | 1006 (0.01) | 917 (0.08) | 922 (0.08) | 992 (0.01) |
| 50 | Proposed | 187 (0.07) | 189 (0.06) | 191 (0.05) | 196 (0.02) | 488 (0.02) | 482 (0.04) | 478 (0.04) | 472 (0.06) | 942 (0.06) | 939 (0.06) | 932 (0.07) | 950 (0.05) |
| | TR-1 | 478 (1.39) | 366 (0.83) | 272 (0.36) | 217 (0.09) | 1221 (1.44) | 1104 (1.21) | 702 (0.40) | 636 (0.27) | 2610 (1.61) | 2229 (1.23) | 1422 (1.42) | 1240 (0.24) |
| | TR-2 | 439 (1.20) | 385 (0.93) | 258 (0.29) | 230 (0.15) | 1203 (1.41) | 1008 (1.02) | 669 (0.34) | 580 (0.16) | 2724 (1.72) | 2426 (1.43) | 1236 (0.24) | 1107 (0.11) |
| | TR-3 | 80.0 (0.60) | 88.0 (0.56) | 127 (0.37) | 144 (0.28) | 157 (0.69) | 185 (0.63) | 296 (0.41) | 341 (0.32) | 255 (0.75) | 324 (0.68) | 558 (0.44) | 669 (0.33) |
| | TR-4 | 192 (0.04) | 194 (0.03) | 193 (0.04) | 202 (0.01) | 502 (0.00) | 483 (0.03) | 484 (0.03) | 518 (0.04) | 911 (0.09) | 990 (0.01) | 1001 (0.00) | 979 (0.02) |
| 100 | Proposed | 199 (0.00) | 198 (0.01) | 191 (0.05) | 196 (0.02) | 498 (0.00) | 483 (0.03) | 475 (0.05) | 491 (0.02) | 971 (0.03) | 951 (0.05) | 926 (0.07) | 931 (0.07) |
| | TR-1 | 365 (0.83) | 288 (0.44) | 234 (0.17) | 201 (0.01) | 947 (0.89) | 696 (0.39) | 598 (0.20) | 512 (0.02) | 2148 (1.15) | 2020 (1.02) | 1151 (0.15) | 967 (0.03) |
| | TR-2 | 338 (0.69) | 308 (0.54) | 245 (0.23) | 216 (0.08) | 996 (0.99) | 806 (0.61) | 580 (0.16) | 535 (0.07) | 1975 (0.98) | 1671 (0.67) | 1074 (0.07) | 1040 (0.04) |
| | TR-3 | 77.5 (0.61) | 88.0 (0.56) | 128 (0.36) | 146 (0.27) | 151 (0.70) | 180 (0.64) | 298 (0.40) | 344 (0.31) | 260 (0.74) | 333 (0.67) | 571 (0.43) | 659 (0.34) |
| | TR-4 | 205 (0.03) | 197 (0.02) | 206 (0.03) | 203 (0.02) | 523 (0.05) | 514 (0.03) | 468 (0.06) | 529 (0.06) | 1097 (0.10) | 996 (0.00) | 978 (0.02) | 904 (0.10) |

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Table 2: Estimated $AARL_0$ (with $\delta$ in parentheses) of the proposed control scheme, the TR-1 method, the TR-2 method, the TR-3 method, and the TR-4 method when $m_0 = 20$ and $\lambda = 0.1$

| | $\alpha$ [$ARL_0$] | 0.005 [200] | | | 0.002 [500] | | | 0.001 [1000] | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\theta_0$ | $n_t$ | 20 | 200 | 2000 | 20 | 200 | 2000 | 20 | 200 | 2000 |
| 0.1 | Proposed | 188 (0.06) | 186 (0.07) | 191 (0.05) | 471 (0.06) | 484 (0.03) | 465 (0.07) | 940 (0.06) | 953 (0.05) | 960 (0.04) |
| | TR-1 | 501 (1.51) | 498 (1.49) | 532 (1.66) | 1391 (1.78) | 1384 (1.77) | 1494 (1.99) | 2911 (1.91) | 2993 (1.99) | 2985 (1.99) |
| | TR-2 | 521 (1.61) | 529 (1.65) | 524 (1.62) | 1463 (1.93) | 1418 (1.84) | 1450 (1.90) | 2821 (1.82) | 2904 (1.90) | 2816 (1.82) |
| | TR-3 | 26.0 (0.87) | 88.4 (0.56) | 150 (0.25) | 42.1 (0.92) | 184 (0.63) | 355 (0.29) | 57.9 (0.94) | 315 (0.69) | 683 (0.32) |
| | TR-4 | 182 (0.09) | 183 (0.09) | 191 (0.05) | 467 (0.07) | 513 (0.03) | 483 (0.03) | 905 (0.10) | 1073 (0.07) | 1099 (0.10) |
| 1 | Proposed | 190 (0.05) | 184 (0.08) | 193 (0.04) | 470 (0.06) | 498 (0.00) | 495 (0.01) | 955 (0.04) | 956 (0.04) | 972 (0.03) |
| | TR-1 | 497 (1.49) | 510 (1.55) | 542 (1.71) | 1356 (1.71) | 1346 (1.69) | 1233 (1.47) | 2863 (1.86) | 3113 (2.11) | 3190 (2.19) |
| | TR-2 | 528 (1.64) | 517 (1.59) | 539 (1.70) | 1423 (1.85) | 1419 (1.84) | 1452 (1.90) | 2830 (1.83) | 2918 (1.92) | 2862 (1.86) |
| | TR-3 | 87.8 (0.56) | 152 (0.24) | 181 (0.10) | 183 (0.63) | 346 (0.31) | 453 (0.09) | 326 (0.67) | 658 (0.34) | 891 (0.11) |
| | TR-4 | 194 (0.03) | 195 (0.03) | 185 (0.075) | 522 (0.04) | 484 (0.03) | 501 (0.00) | 933 (0.07) | 1032 (0.03) | 927 (0.07) |
| 5.0 | Proposed | 190 (0.05) | 185 (0.08) | 188 (0.06) | 490 (0.02) | 480 (0.04) | 488 (0.03) | 964 (0.04) | 967 (0.03) | 963 (0.04) |
| | TR-1 | 514 (1.57) | 526 (1.63) | 532 (1.66) | 1419 (1.84) | 1286 (1.57) | 1347 (1.69) | 2930 (1.93) | 2919 (1.92) | 2883 (1.88) |
| | TR-2 | 541 (1.71) | 493 (1.47) | 527 (1.63) | 1497 (1.99) | 1362 (1.72) | 1423 (1.85) | 2840 (1.84) | 2922 (1.92) | 2930 (1.93) |
| | TR-3 | 135 (0.33) | 172 (0.14) | 196 (0.02) | 304 (0.39) | 429 (0.14) | 472 (0.06) | 583 (0.42) | 833 (0.17) | 909 (0.09) |
| | TR-4 | 193 (0.04) | 217 (0.09) | 194 (0.03) | 485 (0.03) | 523 (0.05) | 542 (0.08) | 1012 (0.01) | 935 (0.07) | 1084 (0.08) |
| 10.0 | Proposed | 194 (0.03) | 190 (0.05) | 192 (0.04) | 478 (0.04) | 487 (0.03) | 481 (0.04) | 944 (0.06) | 960 (0.04) | 963 (0.04) |
| | TR-1 | 518 (1.59) | 505 (1.53) | 497 (1.49) | 1284 (1.57) | 1315 (1.63) | 1300 (1.60) | 3008 (2.01) | 2844 (1.84) | 3007 (2.01) |
| | TR-2 | 491 (1.46) | 521 (1.61) | 509 (1.55) | 1481 (1.96) | 1346 (1.69) | 1474 (1.95) | 2704 (1.70) | 2886 (1.89) | 2783 (1.78) |
| | TR-3 | 153 (0.24) | 181 (0.10) | 198 (0.01) | 359 (0.28) | 442 (0.12) | 484 (0.03) | 683 (0.32) | 906 (0.09) | 931 (0.07) |
| | TR-4 | 190 (0.05) | 188 (0.06) | 206 (0.03) | 493 (0.01) | 471 (0.06) | 531 (0.06) | 1094 (0.09) | 1043 (0.04) | 911 (0.09) |

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Table 3: Estimated average IC ARL of the proposed control scheme with the time-varying sample sizes when $\lambda = 0.1$, $m_0 = 20$ and $\alpha = 0.005$. Absolute relative error from $\text{ARL}_0 = 200$ is in parentheses.

| Scenario | $\theta_0$ | | | |
|---|---|---|---|---|
| | 0.1 | 1 | 5 | 10 |
| A | 186 (0.07) | 196 (0.02) | 194 (0.03) | 194 (0.03) |
| B | 190 (0.05) | 195 (0.02) | 194 (0.03) | 195 (0.03) |
| C | 183 (0.09) | 191 (0.05) | 196 (0.02) | 191 (0.05) |



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Table 4: Estimated $AARL_1$'s of the proposed control scheme and the TR-4 method when $\theta_0 = 1$, $n_t = 20$, $m_0 = 20$, and $\alpha = 0.005$.

| $\theta_1$ | $\tau = 21, \lambda = 0.1$ | | $\tau = 41, \lambda = 0.1$ | | $\tau = 21, \lambda = 0.5$ | |
|---|---|---|---|---|---|---|
| | Proposed | TR-4 | Proposed | TR-4 | Proposed | TR-4 |
| 1.025 | 138 | 157 | 126 | 134 | 159 | 166 |
| 1.050 | 92.5 | 101 | 81.1 | 87.1 | 129 | 143 |
| 1.075 | 60.2 | 64.3 | 49.1 | 55.7 | 99.2 | 106 |
| 1.100 | 41.6 | 42.7 | 29.5 | 30.5 | 75.4 | 90.8 |
| 1.125 | 24.8 | 26.6 | 19.2 | 19.5 | 57.6 | 66.4 |
| 1.150 | 16.8 | 17.7 | 14.2 | 15.8 | 38.1 | 43.4 |
| 1.175 | 11.8 | 13.8 | 10.2 | 10.3 | 31.2 | 33.6 |
| 1.200 | 9.52 | 9.94 | 8.78 | 8.76 | 19.7 | 21.4 |
| 1.500 | 2.24 | 2.28 | 2.22 | 2.23 | 1.51 | 1.68 |

Table 5: Estimated AARL (SDARL) of the proposed control scheme when $\lambda = 0.1$.

| $n_t$ | $m_0$ | $\alpha = 0.005$ [$E(\text{ARL}_0) = 200$] $\theta_0 = 0.1$ | $\alpha = 0.005$ [$E(\text{ARL}_0) = 200$] $\theta_0 = 1$ | $\alpha = 0.002$ [$E(\text{ARL}_0) = 500$] $\theta_0 = 1$ |
|---|---|---|---|---|
| 20 | 10 | 191 (20.3) | 192 (18.4) | 499 (37.6) |
| | 20 | 193 (16.8) | 191 (14.4) | 498 (51.5) |
| | 30 | 189 (17.7) | 189 (15.5) | 499 (47.1) |
| | 50 | 190 (16.3) | 192 (19.6) | 484 (34.2) |
| 50 | 10 | 198 (16.3) | 200 (15.4) | 500 (49.1) |
| | 20 | 193 (18.1) | 195 (16.9) | 499 (48.6) |
| | 30 | 190 (17.2) | 191 (19.0) | 482 (48.3) |
| | 50 | 188 (18.4) | 186 (18.9) | 482 (45.4) |
| 200 | 10 | 193 (18.1) | 198 (20.5) | 499 (47.9) |
| | 20 | 193 (17.7) | 195 (18.2) | 498 (35.9) |
| | 30 | 190 (17.2) | 190 (17.9) | 488 (43.9) |
| | 50 | 192 (15.1) | 190 (17.6) | 483 (36.0) |



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Table 6: Estimated $\mathrm{AARL}_0$ (with $\delta$ in parentheses) of the proposed control scheme when $m_0 = 20$ and $\alpha = 0.005$ ($E(\mathrm{ARL}_0) = 200$).

| $N$ | $\lambda = 0.10$ | | | | $\lambda = 0.40$ | | | | $\lambda = 0.80$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\theta_0 = 1$ | | $\theta_0 = 5$ | | $\theta_0 = 1$ | | $\theta_0 = 5$ | | $\theta_0 = 1$ | | $\theta_0 = 5.0$ | |
| | $n_t$ | | $n_t$ | | $n_t$ | | $n_t$ | | $n_t$ | | $n_t$ | |
| | 20 | 200 | 20 | 200 | 20 | 200 | 20 | 200 | 20 | 200 | 20 | 200 |
| 200 | 122 | 124 | 125 | 125 | 99.4 | 98.7 | 97.8 | 99.5 | 89.9 | 91.8 | 92.4 | 91.8 |
| | (0.39) | (0.38) | (0.38) | (0.38) | (0.50) | (0.51) | (0.51) | (0.50) | (0.55) | (0.54) | (0.54) | (0.54) |
| 500 | 187 | 192 | 191 | 189 | 162 | 166 | 165 | 167 | 155 | 157 | 157 | 158 |
| | (0.07) | (0.04) | (0.05) | (0.06) | (0.19) | (0.17) | (0.18) | (0.17) | (0.23) | (0.22) | (0.22) | (0.21) |
| 1000 | 173 | 177 | 174 | 177 | 162 | 160 | 159 | 162 | 154 | 156 | 155 | 158 |
| | (0.14) | (0.12) | (0.13) | (0.12) | (0.19) | (0.20) | (0.21) | (0.19) | (0.23) | (0.22) | (0.23) | (0.21) |
| 2000 | 185 | 187 | 185 | 186 | 176 | 176 | 174 | 174 | 169 | 171 | 173 | 173 |
| | (0.08) | (0.07) | (0.08) | (0.07) | (0.12) | (0.12) | (0.13) | (0.13) | (0.16) | (0.15) | (0.14) | (0.14) |
| 5000 | 192 | 195 | 192 | 192 | 181 | 186 | 182 | 186 | 178 | 181 | 176 | 178 |
| | (0.04) | (0.03) | (0.04) | (0.04) | (0.10) | (0.07) | (0.09) | (0.07) | (0.11) | (0.10) | (0.12) | (0.11) |
| 10000 | 191 | 192 | 194 | 194 | 186 | 191 | 185 | 186 | 184 | 184 | 183 | 188 |
| | (0.05) | (0.04) | (0.03) | (0.03) | (0.07) | (0.05) | (0.08) | (0.07) | (0.08) | (0.08) | (0.09) | (0.06) |



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Table 7: Control limit, charting statistic, and detected status of four methods with respect to the incidence rate of melanoma.2

| Year | Proposed | | | TR-1 | | | TR-2 | | | TR-3 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | CL | CS | S | CL | CS | S | CL | CS | S | CL | CS | S |
| 1995 | 0.244 | 0.087 | IC | 0.490 | 0.087 | IC | 0.249 | 0.087 | IC | 0.398 | 0.089 | IC |
| 1996 | 0.310 | 0.327 | OC | 0.490 | 0.333 | IC | 0.311 | 0.327 | OC | 0.398 | 0.314 | IC |
| 1997 | - | - | - | 0.490 | 0.565 | OC | - | - | - | 0.398 | 0.515 | OC |



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