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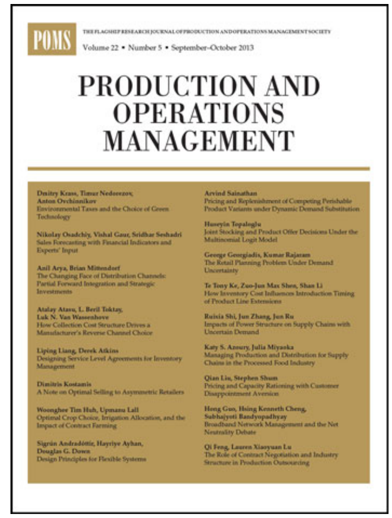

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Managing Supply Chains with Expediting and Multiple Demand Classes

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We consider a periodic-review, single-stage inventory system with multiple demand classes and a fixed replenishment leadtime. Inventory expediting is allowed to alleviate demand-supply mismatches. Priority demands are commonly used in practice to provide differentiated services for customers, and inventory expediting is an effective strategy to improve the service levels of high-priority demands. However, it is challenging to coordinate the inventory ordering, expediting and allocation decisions in supply chains. We partially characterize the structure of the optimal policy. We also derive various monotone properties of the optimal policy with limited sensitivities. Moreover, we show that a state-dependent rationing level policy is optimal for inventory allocation and the optimal rationing levels are in fact independent of the backorder quantity of each demand class. We also show when some simple policies are indeed optimal. Numerically, we illustrate the optimal policy and investigate the performances of three proposed simple heuristics. Finally, we extend the results to more general systems, such as serial systems, systems with convex backordering costs, etc.

Key words: inventory rationing; multiple demand classes; leadtimes; expediting
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1. Introduction

In practice, customers are usually classified into different priorities based on their respective delivery time requirements, contractual relationships, profit margins, etc. For example, customers with

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earlier fulfillment requirements would be naturally assigned with a higher priority; customers with long-term supply contracts usually receive a higher priority from their supplier over regular customers. Even if a firm offers the same price for different customers, the transportation costs may vary for different segments of customers, which results in different profit margins.

Facing customers with different priorities, firms can segment customers into different demand classes with different backordering costs to provide differentiated services (Arslan et al. 2007). To manage the fulfillments of multiple demand classes, inventory rationing is widely adopted: There is a *rationing level* for each demand class such that, when the inventory level is at or below that level, it is optimal not to fulfill demand of the corresponding class and reserve inventory for future demands with higher priorities. Inventory rationing is an increasingly important strategy used for matching costly supply with uncertain demands when customers have different service priorities (Deshpande et al. 2003).

To reduce the backordering costs, especially for high-priority customers, firms may resort to inventory expediting from multiple locations in their supply chains (Kim et al. 2015), i.e., expediting the delivery of partial or complete orders through either overtime work or premium delivery such as air freight. Inventory expediting is also commonly adopted to reduce supply chain costs for the equipment manufacturers and the service parts industry. Özsen and Thonemann (2015) indicate that, in a recent survey, inventory managers from 28 companies respond that they expedite orders to avoid backorders among 32 surveyed companies in the European divisions of manufacturing companies.

Our research is partially motivated by a consulting project for a cross-border e-commerce firm in China, where priority demands, inventory allocation and expediting are integrally implemented. The firm specializes in selling imported goods, e.g., foods and wines, from a global supplier in Australia. On its online platform, different customers may have different delivery time requirements. For example, they can choose one-day delivery or three-day delivery. To reduce transportation costs, the firm has a central warehouse in a free-trade zone to lower import tax and several local warehouses in different locations. The local warehouses replenish inventory from the warehouse in the free-trade zone and in turn it replenishes inventory from the global supplier by sea freight. When the inventory of a local warehouse is low, the firm may resort to air freight to expedite inventory from the central warehouse or even the global supplier directly in order to maintain high service levels. The challenge of the firm is to decide the ordering quantities from the global supplier and the inventory allocation and expediting strategies in order to minimize the total supply chain cost.

Partially motivated by the consulting project, we investigate the coordination of inventory ordering, expediting and allocation decisions to minimize the total cost of a supply chain with multiple demand classes, inventory expediting and a fixed replenishment leadtime. We assume a linear replenishment cost and there is no limit on the ordering quantity in each period. Demands in different periods are independent and unfulfilled demands are fully backlogged. Leftover inventory is carried over to the next period. Demand classes are differentiated by their respective unit backordering costs. We assume that the firm can expedite inventory from any leadtime position of the supply chain. The supply chain without inventory expediting is a special case of our model.

Notice that the interactions among the inventory ordering, expediting and allocation are complex, especially in the presence of a fixed replenishment leadtime. With a fixed leadtime, the optimal policy may depend on the whole pipeline inventories as, in addition to the on-hand inventory, the demands in each period can be fulfilled by the pipeline inventories through expediting. Our objective is three-fold: (1) to provide the characterization of the structure of the optimal policy for inventory ordering, expediting and allocation; (2) to understand how backordering, expediting and holding costs affect the optimal policy; (3) to identify when a simple policy can be optimal, which provides useful benchmarks on designing effective heuristics.

The main results of this paper are summarized as follows. First, under a convex expediting cost structure, we show that it is optimal to sequentially fulfill demands with higher unit backordering costs first and expedite inventory from lower leadtime positions first. Second, through a novel transformation of state variables, we show the $L^\natural$-convexity of the value functions based on a new preservation property. The $L^\natural$-convexity can be used to design effective approximation algorithms (Chen et al. 2014).

Based on the above two results, we then partially characterize the structure of the optimal policy for inventory ordering, expediting and allocation. Specifically, the optimal ordering policy is a state-dependent base stock policy, where the base stock level in each period depends on the backorder quantities of different demand classes and the inventory levels of different leadtime positions. The optimal expediting policy can be described by a state-dependent threshold policy: There exists a state-dependent threshold for each leadtime position; if the on-hand inventory level is below the threshold, it is optimal to expedite inventory up to that threshold; otherwise it is optimal not to expedite from that leadtime position. The optimal allocation policy can be described by a state-dependent rationing level policy: There exists a state-dependent rationing level for each demand class; if the backorder quantity is above the threshold, it is optimal to fulfill the demand and reduce the backorder quantity to that threshold.

We also obtain a series of monotone properties of the optimal policy. These monotone properties not only facilitate the computation of the optimal policy but also enhance our understandings on how to manage such a supply chain. More interestingly, based on these monotone properties, we show that the optimal rationing levels are *independent* of backorder quantities of different demand classes. In other words, we can ignore the demand state in obtaining the optimal rationing levels, which significantly reduces the computational time. This result can be viewed as a non-trivial extension on the optimality of a state-independent rationing level policy in Topkis (1968), where no replenishment is allowed during the planning horizon, to the inventory systems with replenishments and fixed leadtimes. In fact, our result also implies that, with zero replenishment leadtime, a state-independent rationing level policy is optimal in the presence of replenishments.

Moreover, we provide conditions under which simple *state-independent* policies turn out to be optimal. These simple policies may be used as heuristics even if the optimality conditions are violated. Inspired by the above results, we also propose three simple heuristics and numerically investigate the performance gaps of the three heuristics against the optimal policy. The numerical studies reveal the value of inventory expediting and the importance of managing inventory expediting appropriately. The studies also show that, under many scenarios, the heuristics are effective. Finally, in Appendix B, we show that the main results of the basic model are robust to systems with convex backordering costs, Markov modulated demands, fixed ordering intervals, stochastic sequential leadtimes, and multiple stages.

The remainder of this paper is organized as follows. In Section 2, we review the related literature. In Section 3, we describe our model and formulation. The structural results of the optimal policy are presented in Section 4. In Section 5, we consider several special cases under which some simple policies are indeed optimal. We then numerically illustrate the optimal policy and investigate the performances of three proposed heuristics in Section 6. Finally, we provide some concluding remarks in Section 7. All the proofs are relegated in Appendix A (noting that some intuitive derivations may be omitted in the proof).

2. Related Literature

Our paper is related to two streams of literature, one dealing with *inventory rationing with multiple customer classes* and the other with *inventory expediting*. The concept of inventory rationing is first introduced by Veinott (1965) who proposes a state-independent critical level policy to allocate the on-hand inventory to different demand classes. He also analyzes under what circumstance a myopic ordering policy is optimal. However, the myopic policy in general is not optimal under his setting. Topkis (1968) shows the optimality of a state-independent rationing level policy when there is no replenishment during the planning horizon. Evans (1968) and Kaplan (1969) derive the same results as in Topkis (1968) for two demand classes under the lost-sales setting and the backordering setting, respectively. These early works focus on the structure of optimal policies for systems with multiple demand classes but no replenishment during the planning horizon. The applicability of these results is rather limited due to no replenishment. In contrast, we consider a periodic-review inventory system with multiple demand classes and a fixed replenishment leadtime. With replenishments during the planning horizon, the state-independent rationing level policy and the simple base stock policy may no longer be optimal as in general the optimal ordering policy depends on the pipeline inventories and demand state.

More recently, there has been some work on systems with multiple demand classes and fixed setup costs. Frank et al. (2003) consider a system with two demand classes, stochastic and deterministic. Stochastic demand is lost if not satisfied immediately and deterministic demand must be fulfilled upon arrival. They show the optimality of a state-dependent (s,S) ordering policy and propose a simple heuristic. Chen et al. (2010) derive similar results for a system with two classes of stochastic demands under the backordering setting. They also partially depict the structural properties of rationing based on the subconvexity of cost functions. Zhou and Zhao (2010) consider that a part of demand classes follows the lost-sales setting while the others follow the backordering setting. They indicate that a partially state-dependent (s,S) policy for the ordering decisions and a state-dependent priority policy for the allocation decisions are optimal. Different from this stream of literature, we do not incorporate the fixed setup costs but consider a fixed replenishment leadtime and inventory expediting.

For continuous-review systems, there is a stream of literature focusing on the analysis of optimal policies for make-to-stock production systems. Ha (1997a) considers inventory rationing in an $M/M/1$ make-to-stock queue with multiple demand classes and lost sales. He shows that a lot-for-lot policy for production and a stationary critical level policy for rationing are optimal. Ha (1997b) derives similar results for a system with two demand classes under the backordering setting. The backordering model is then extended by de Véricourt et al. (2000, 2002) to multiple demand classes. Gayon et al. (2009) use advance demand information in the context of a production-inventory system with multiple demand classes and show the structures of the optimal production policy and the optimal rationing policy. Our work differentiates from this stream of literature in the sense that we consider the model under the periodic-review setting with inventory expediting and a fixed replenishment leadtime.

Due to the difficulty of applying state-dependent optimal policies in practice, a stream of literature focuses on developing heuristics and evaluating their performances. Cohen et al. (1988) consider an (s, S) policy and propose effective approximations for an inventory system with two demand classes under the lost-sales setting. Möllering and Thonemann (2008) investigate an inventory system with two demand classes under the backordering setting and assume a state-independent rationing level policy. They develop an approach to evaluate the optimal rationing levels. The above studies consider the periodic-review setting and there are some other studies focusing on the continuous-review setting. Nahmias and Demmy (1981) consider a two-class system with at most one outstanding order at any time. They analyze the system based on a (Q, R) inventory replenishment policy and a fixed rationing level policy. Dekker et al. (1998) analyze a special case with $Q = 1$ and Deshpande et al. (2003) generalize the results to the system without the restriction on the number of outstanding orders. Deshpande and Cohen (2005) then extend the model in Deshpande et al. (2003) to multiple demand classes under a threshold clearing policy for the backorders. Fadıloğlu and Bulut (2005) analyze the same setting as in Deshpande et al. (2003) by using an embedded Markov chain approach and a one-for-one inventory replenishment policy. Arslan et al. (2007) consider essentially the same threshold clearing policy as Deshpande et al. (2003). They assume a continuous-review (Q, R) replenishment policy and a critical level policy for inventory rationing among multiple demand classes. All the above studies consider the backordering setting. In a (Q, R) system with lost sales, Melchiors et al. (2000) develop an approach to evaluate the inventory policy exactly by using a constant rationing level policy. Our study, however, mainly focuses on analyzing the structure of the optimal policy though we also numerically investigate the performances of some heuristics.

On the other hand, inventory expediting has been widely observed in practice and proven to be an effective strategy to reduce supply chain costs. As indicated by Mamani and Moinzadeh (2014), expedited orders are different from emergency orders in the sense that the former effectively change the leadtime for an in-transit order. We thus focus on reviewing the literature of inventory expediting below. For the literature on emergency orders, one may refer to, e.g., Moinzadeh and Nahmias (1988) and Moinzadeh and Schmidt (1991). Our paper is closely related to the literature on periodic-review inventory models with expediting. Lawson and Porteus (2000) are among the first to consider a serial multi-echelon system that allows expediting between two consecutive echelons. By assuming linear expediting costs, they show that a top-down echelon base stock policy is optimal. Muharremoğlu and Tsitsiklis (2003) analyze the optimal ordering policy under a similar model of Lawson and Porteus (2000) by allowing a more general expediting structure. Huggins and Olsen

(2003) describe the optimal policy for a two-echelon system where orders can be expedited from upstream and from an outside supplier directly. For continuous-review inventory systems, Kouvelis and Tang (2012) examine the use of an expediting service in dealing with a stochastic leadtime and deterministic demands. They characterize the optimal expediting policy and explore the impact of the expediting option on the replenishment order. Mamani and Moinzadeh (2014) assume that in-transit orders are shipped through multiple intermediate stages and can be expedited from any stage to the destination. They propose an expediting policy and focus on the effects of shipment network design, such as expediting hub number and location, on the performance of the retailer or the supply chain. With the same expediting structure as in Lawson and Porteus (2000), Angelus and Özer (2016) consider the expediting of stock in an assembly system and analyze its optimal policy. Our work differentiates from the above studies in the sense that we can expedite inventory from any upstream stages directly to the most downstream stage and we deal with multiple demand classes. We characterize the optimal policy for a joint inventory ordering, allocation and expediting decision.

In particular, different from the assumption of expediting between two consecutive echelons in Lawson and Porteus (2000) and Muharremoğlu and Tsitsiklis (2003), Kim et al. (2015) allow outstanding orders to be expedited from the outside supplier or any intermediate stage to the most downstream stage directly and derive an optimal expediting level policy for sequential systems. The expediting structure in Kim et al. (2015) is similar to ours. However, they consider a system with a single demand class while we consider a system with multiple demand classes. Under our setting, we have to consider the interactions among inventory ordering, allocation and expediting. There are also some related studies focusing on effective heuristics. For example, Özsen and Thonemann (2015) propose a threshold expediting policy and then determine the optimal parameters for the simplified model where only variable expediting costs are considered. Tao et al. (2017) examine a two-replenishment-mode model, namely regular mode and expediting mode. They propose an innovative ordering policy (S, e) where S is the order-up-to level and expediting happens when the inventory level drops to or below a certain level e.

Although there is an extensive literature on inventory systems with multiple demand classes and inventory systems with inventory expediting, the two features — multiple demand classes and inventory expediting — are usually considered separately. We are among the first to analyze the inventory system with multiple demand classes and inventory expediting simultaneously.

To analyze our problem, we use the L$^\natural$-convexity, which is developed by Murota (2003, 2005) and Zipkin (2008). The L$^\natural$-convexity has been widely applied for structural analysis in inventory models as it implies strong convexity and submodularity. Related studies include Lu and Song (2005) for assemble-to-order systems, Zipkin (2008) for lost-sales models, Huh and Janakiraman (2010) for a serial lost-sales model, and Pang et al. (2012) for a joint inventory-pricing control problem with a fixed leadtime. In particular, Lu and Song (2005) show the L$^\natural$-convexity in the original state variables, whereas Zipkin (2008) and Pang et al. (2012) show it in the transformed state variables (i.e., different transformations of the inventory level and outstanding orders). Huh and Janakiraman (2010) use bi-echelon inventories to show the L$^\natural$-convexity. Moreover, Gong and Chao (2013) adopt the L$^\natural$-convexity to analyze inventory systems with remanufacturing; Chen et al. (2014) apply the L$^\natural$-convexity to characterize the optimal pricing and ordering policy for perishable products; Li and Yu (2014) show that the L$^\natural$-convexity can be transformed into multimodularity, and apply it in several inventory problems, including perishable inventory with clearance. In this study, we utilize a novel transformation for the state variables of inventory systems with multiple demand classes and expediting. We show that L$^\natural$-convexity is preserved under a special minimization operator. Then, based on the L$^\natural$-convexity, we find that the optimal rationing levels are independent of backorders.

3. Model and Formulation

We consider a single-product, single-stage and periodic-review inventory system with multiple demand classes and a fixed replenishment leadtime l under a finite planning horizon with T periods. The system replenishes from an outside supplier with ample stock. Hence, there is no limit on how much we can order in each period. We allow expediting in-transit inventory from any leadtime positions. The quantity of inventory that can be expedited from any leadtime position is limited by its local inventory.

There are $n+1$ demand classes indexed by $0,\cdots,n$, which are segmented by their respective unit backordering costs. In each period t, $t=1,\cdots,T+l$, the demand of each class is random and nonnegative, denoted by $D_{j,t}$ for $j=0,\cdots,n$. Demands in the same period can be correlated but demands across different periods are independent and identically distributed (note that the identical distribution is assumed for the ease of presentation; our results are robust when demand distributions are non-identical across different periods). For example, the demands in the same period can result from endogenous customer choices while demands in different periods correspond to different "generations" of customers. Unfulfilled demands are fully backlogged and leftover inventory is carried over to the next period. There are a unit holding cost h for leftover inventory, a unit ordering cost c and a unit backordering cost b_j for backorders of class j, $j=0,\cdots,n$. Without loss of generality, we assume that $b_0>b_1>\cdots>b_n$, i.e., class j demand has a higher unit backordering cost over class $j+1$ demand for any $j \in \{0, \cdots, n-1\}$. If the on-hand inventory is not sufficient to fulfill demands, we can expedite inventory from any leadtime positions $1, \cdots, l$ directly to the most downstream stage (which is referred to as leadtime position 0) with a unit expediting cost s_i, $i = 1, \cdots, l$. We assume that the difference between the unit expediting costs of two adjacent leadtime positions is nondecreasing to ensure the optimality of the sequential expediting property.

Assumption 1 **(Convex Expediting Cost Structure)**. *The unit expediting costs satisfy the following property:* $s_{i+1} - s_i \geq s_i - s_{i-1} \geq 0$ *for* $i = 1, \cdots, l-1$ $(s_0 \equiv 0)$.

This assumption has also been adopted by Kim et al. (2015) for the same purpose. Essentially, we require that the unit expediting cost is increasingly convex in leadtime position. It is reasonable in the following practical situations: In manufacturing, overtime may be sufficient if we expedite from a lower leadtime position with semi-manufactures. However, if we expedite from a higher leadtime position with raw materials, more efforts and coordination are required to transform the raw materials to the end product. In transportation, we can choose low-cost transportation modes like highway and railway if we expedite from a lower leadtime position, e.g., a local warehouse, while we may resort to the high-cost air freight if we expedite from a higher leadtime position, e.g., a global manufacturing facility. Finally, we allow a discount factor $\beta \in (0, 1]$.

Remark 1. (1) Our model generalizes the following special cases: If inventory expediting is allowed or effective only at lower leadtime positions $1, \cdots, k$, then $s_{k+1}, \cdots, s_l$ are sufficiently large so that no expediting would be implemented; if expediting is not allowed in the whole system, then $s_1, \cdots, s_l$ are sufficiently large and the problem reduces to the standard inventory problem with multiple demand classes. (2) Our results are robust to the expediting from an outside supplier with a unit cost s_{l+1} as long as $s_{l+1} - s_l \geq s_l - s_{l-1} \geq 0$ in addition to Assumption 1.

The sequence of events in each period is as follows: (1) At the beginning of each period, the order due in this period arrives; (2) An order is placed before demand realization; (3) Demands realize and the system manager makes the expediting and allocation decisions based on the system state (including the backorder quantity of each class and the inventory levels of different leadtime positions); (4) Finally, all costs are charged at the end of the period.

In each period t $(t = 1, \cdots T + l)$, the system manager makes an ordering decision first before demand realization and then makes inventory expediting and allocation decisions to minimize the expected cost (including ordering, expediting, backordering and holding costs) from the period and onward. For the expediting decision, the system manager decides where to expedite from and how much to expedite. For the inventory allocation problem, the primary issue is how to allocate on-hand inventory to fulfill backorders of different demand classes. Alternatively, the inventory allocation problem can be equivalently stated as an inventory rationing problem, i.e., the primary issue is to trade off fulfilling lower-priority demands now versus reserving the inventory for higher-priority demands in the future. In the sequel, we use the terms *inventory allocation policy* and *inventory rationing policy* interchangeably.

For the ease of understanding, we provide the list of notations used in this paper below. For $i=0,\cdots,l$ and $j=0,\cdots,n$, we define

b_j=unit backordering cost of demand class j;

s_i=unit expediting cost from leadtime position i to leadtime position 0;

h=unit holding cost;

c=unit ordering cost;

$D_{j,t}$= amount of class j demand realized in period t;

$w_{j,t}$=backorder quantity of demand class j at the beginning of period t;

$z_{j,t}=-\sum_{k=j}^{n} w_{k,t}$=negatively aggregated backorder quantity from class j to class n;

$x_{0,t}$=amount of on-hand inventory at the beginning of period t;

$x_{i,t}=q_{t-l+i}$=amount of the order placed in period $t-l+i$, $i\geq 1$;

$a_{j,t}$=amount of inventory allocated to demand class j in period t;

$o_{i,t}$=amount of expedited inventory from leadtime position i in period t;

$v_{i,t}=\sum_{k=0}^{i} x_{k,t}+z_{0,t}$=amount of net inventory level at leadtime position i at the beginning of period t.

We may drop the subscript t of the variables for convenience. We also use the notions $[\phi]^+=\max\{\phi,0\}$, $\phi\vee\zeta=\max\{\phi,\zeta\}$, $\phi\wedge\zeta=\min\{\phi,\zeta\}$, $\zeta\vee\Phi=(\zeta\vee\phi_1,\cdots,\zeta\vee\phi_n)$ for $\Phi=(\phi_1,\cdots,\phi_n)$ and assume $\sum_{k=i}^{j}\zeta=0$ for any variable ζ when $i>j$.

Let $\boldsymbol{w}=(w_0,\cdots,w_n)$, $\boldsymbol{x}=(x_0,\cdots,x_{l-1})$, $\boldsymbol{D}=(D_0,\cdots,D_n)$, $\bar{\boldsymbol{x}}=(x_0,x_1,\cdots,x_l)$, $\boldsymbol{a}=(a_0,\cdots,a_n)$, $\boldsymbol{o}=(o_1,\cdots,o_l)$, and $\bar{\boldsymbol{w}}=\boldsymbol{w}+\boldsymbol{D}$. Let $f_t(\boldsymbol{w},\boldsymbol{x})$ be the optimal cost of the system from period t and onward. In each period, the system manager first makes the ordering decision and then, upon demand realization, makes the expediting and allocation decisions. For notational simplicity, let x_l be the ordering quantity in each period. Then, the dynamic recursion is given as follows:

$$f_t(\boldsymbol{w},\boldsymbol{x})=\min_{x_l\geq 0}\mathbb{E}[cq+g_t(\boldsymbol{w}+\boldsymbol{D},\boldsymbol{x},x_l)], \tag{1}$$

and $g_t(\boldsymbol{w}+\boldsymbol{D},\boldsymbol{x},x_l)=g_t(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})$ with

$$g_t(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=\min_{\boldsymbol{a},\boldsymbol{o}\in\mathcal{A}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})}\left[\sum_{j=0}^{n} b_j(w_j+D_j-a_j)+h\left(x_0+\sum_{i=1}^{l} o_i-\sum_{j=0}^{n} a_j\right)+\sum_{i=1}^{l} s_i o_i+\beta f_{t+1}\Big(\bar{\boldsymbol{w}}-\boldsymbol{a},\right.$$

$$x_0 + x_1 - \sum_{j=0}^{n} a_j + \sum_{i=2}^{l} o_i, x_2 - o_2, \cdots, x_l - o_l \Big) \Big], \tag{2}$$

where

$$\mathcal{A}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) = \left\{ \boldsymbol{a}, \boldsymbol{o} : 0 \le a_j \le w_j + D_j,\ j = 0, \cdots, n;\ \sum_{j=0}^{n} a_j \le x_0 + \sum_{k=1}^{l} o_k,\ 0 \le o_i \le x_i,\ i = 1, \cdots, l \right\}.$$

The feasible region $\mathcal{A}(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}})$ indicates that the total allocation quantity cannot exceed the on-hand inventory after expediting, the allocation to each demand class cannot exceed its backorder quantity, and the expediting quantity from leadtime position i cannot exceed its inventory level x_i. In the terminal period $T + l + 1$, $f_{T+l+1}(\boldsymbol{w}, \boldsymbol{x}) \equiv 0$ for any $(\boldsymbol{w}, \boldsymbol{x})$ as we assume that there is no salvage value for leftover inventory after $T + l$. Note that, the decisions in equation (2) are made after the demand realization. However, the formulas in equations (1) and (2) are too complex to be analyzed as they involve many state and decision variables.

4. Structural Analysis

In this section, we first present a preliminary result on the L$^\natural$-convexity to facilitate the structural analysis and then show the structure of the optimal policy for inventory ordering, expediting and allocation of our problem.

4.1. Preliminary Result

Let $\boldsymbol{e}$ be the unit vector with all the elements being 1, $\boldsymbol{e}_i$ be the unit vector with the i-th element being 1 and all the others being 0, and $V \subseteq \mathbb{R}^n$ be a polyhedron that forms a sublattice. Following Murota (2000, 2003) and Zipkin (2008), we say that a function $f : V \to \mathbb{R}$ is L$^\natural$-convex if the function $\psi(\boldsymbol{y}, \eta) = f(\boldsymbol{y} - \eta \boldsymbol{e})$ is submodular on $\{(\boldsymbol{y}, \eta) | \boldsymbol{y} \in V, \eta \in \mathbb{R}, \boldsymbol{y} - \eta \boldsymbol{e} \in V\}$. Note that the L$^\natural$-convexity implies ordinary convexity, submodularity and diagonal dominance, which in turn imply strong convexity (see Zipkin 2008 for more details). We shall utilize this property in the following analysis.

Below, we show that the L$^\natural$-convexity can be preserved under a special minimization operator. This property is useful to analyze the inventory expediting and allocation policy in our problem.

Lemma 1. *Consider a vector $\boldsymbol{x} = (x_1, \cdots, x_{n+m})$ such that $0 \equiv x_0 \ge x_1 \ge \cdots \ge x_n \le x_{n+1} \le \cdots \le x_{n+m}$. Suppose that $f(\boldsymbol{x})$ is L$^\natural$-convex and for the constants γ_k, $k = 1, \cdots, n + m$,*

$$\tilde{f}(\boldsymbol{x}) = f(\boldsymbol{x}) + \sum_{k=1}^{n+m} \gamma_k x_k$$

is nondecreasing in $x_1, \cdots, x_{n-1}$ and $x_{n+1}, \cdots, x_{n+m-1}$. Also, define

$$H_{ij}(\boldsymbol{x}) = \min_{(y_i, t_{n+j}) \in \mathcal{B}_{ij}} [\tilde{f}(x_1, \cdots, x_{i-1}, \overbrace{y_i, \cdots, y_i}^{n-i+1}, \overbrace{t_{n+j}, \cdots, t_{n+j}}^{j-1}, x_{n+j}, \cdots, x_{n+m}) + \zeta_t t_{n+j} + \zeta_y y_i],$$

where the feasible region of (y_i, t_{n+j}) *is* $\mathcal{B}_{ij} = \{y_i, t_{n+j} | x_i \le y_i \le \min\{x_{i-1}, t_{n+j}\},\ x_{n+j-1} \le t_{n+j} \le x_{n+j}\}$ *and* ζ_t*,* ζ_y *are constants. Then, the function* $g(\boldsymbol{x}) = \min_{\substack{i=1,\cdots,n,\\ j=1,\cdots,m}} H_{ij}(\boldsymbol{x})$ *is also* $L^\natural$*-convex.*

Lemma 1 can be viewed as one of our technical contributions. It can be potentially used in, e.g., dynamic matching problems.

4.2. Analysis and Results

We first show that under the optimal policy, there exist the sequential properties for the inventory allocation and expediting as follows.

Lemma 2. *Under the optimal policy, in each period* $t = 1, \cdots, T + l$*,*

(1) Class k *demand is fulfilled only if there is no unfulfilled demand of class* j *for all* $0 \le j < k \le n$*, i.e., it is optimal to fulfill higher-priority demands first.*

(2) It is optimal to fulfill demands with on-hand inventory first. If expediting is necessary, inventory can be expedited from leadtime position i*,* $i \in \{1, \cdots, l\}$*, only if there is no inventory at leadtime positions* $0, \cdots, i-1$*, i.e., it is optimal to expedite inventory from lower leadtime positions first.*

Lemma 2 directly implies the following corollary.

Corollary 1. *Under the optimal policy, for any state* $(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}})$*, if* $a_j > 0$*, then we must have* $a_k = w_k + D_k$ *for all* $k = 0, \cdots, j-1$*, i.e., it is optimal to fulfill all the demands of classes* $0, \cdots, j-1$ *before we fulfill class* j *demand in each period. Similarly, under the optimal policy, if* $o_i > 0$*, then we must have* $o_k = x_k$ *for all* $k = 1, \cdots, i-1$*.*

Based on the sequential properties in Lemma 2, we adopt the following state transformation to investigate the structure of the optimal policy:

$$z_j = -\sum_{k=j}^{n} w_k, \quad j = 0, \cdots, n, \tag{3}$$

$$v_i = z_0 + \sum_{k=0}^{i} x_k, \quad i = 0, \cdots, l-1. \tag{4}$$

Clearly, we have $0 \ge z_n \ge z_{n-1} \ge \cdots \ge z_0 \le v_0 \le v_1 \le \cdots \le v_{l-1}$. Note that z_j is the negatively aggregated backorder quantity of classes $j, \cdots, n$, and v_i is the (net) inventory level at leadtime position i. We define z_j's so that, based on Lemma 2, under the optimal policy we sequentially change the variables $z_0, \cdots, z_n$ (see equation (9)). This state transformation is critical to show the $\mathrm{L}^\natural$-convexity of the value functions. Moreover, as we shall show in Section 5, this state transformation can lead to the decomposition of value functions for some special cases. For the ease of exposition, let $\boldsymbol{z} = (z_n, \cdots, z_0)$, $\boldsymbol{v} = (v_0, \cdots, v_{l-1})$ and $z_{n+1} \equiv 0$.

Suppose that the order-up-to level is v_l, then after demand realization the state transits to $\bar{\boldsymbol{z}} = (\bar{z}_n, \cdots, \bar{z}_0)$ and $\bar{\boldsymbol{v}} = (\bar{v}_0, \cdots, \bar{v}_{l-1}, \bar{v}_l)$ such that

$$\bar{z}_j = z_j - \sum_{k=j}^{n} D_k, \quad j = 0, \cdots, n,$$
$$\bar{v}_i = v_i - \sum_{k=0}^{n} D_k, \quad i = 0, \cdots, l,$$

before inventory expediting and allocation. Furthermore, suppose that the expediting policy is $\boldsymbol{o}$ and the allocation policy is $\boldsymbol{a}$. Then, at the beginning of the next period, the system state is $\boldsymbol{z}_+ = (z_{n+}, \cdots, z_{0+})$ and $\boldsymbol{v}_+ = (v_{0+}, \cdots, v_{(l-1)+})$ such that

$$z_{j+} = \bar{z}_j + \sum_{k=j}^{n} a_k, \quad j = 0, \cdots, n, \tag{5}$$
$$v_{(i-1)+} = \bar{v}_i + \sum_{k=i+1}^{l} o_k, \quad i = 1, \cdots, l. \tag{6}$$

Under the transformed state, the variables $v_0, \cdots, v_{l-1}$ are independent of the allocation policy while the variables $z_0, \cdots, z_n$ are independent of the expediting policy. This property of the transformed state facilitates our analysis in the sense that we can investigate the inventory expediting and allocation decisions separately based on two different sets of variables.

Define the cost parameters as follows:

$$\begin{cases} \hat{s}_i = s_i - s_{i+1}, i = 0, \cdots, l-1, \\ \hat{b}_j = b_{j-1} - b_j, j = 1, \cdots, n, \\ \hat{b}_0 = -b_0. \end{cases}$$

Then, the dynamic recursion of our problem under the transformed state is given as

$$\bar{f}_t(\boldsymbol{z}, \boldsymbol{v}) = \min_{v_l \geq v_{l-1}} \mathbb{E}\left[c(v_l - v_{l-1}) + \bar{g}_t\left(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, (\boldsymbol{v}, v_l) - \sum_{k=0}^{n} D_k \boldsymbol{e}\right)\right], \tag{7}$$

and $\bar{g}_t\left(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, (\boldsymbol{v}, v_l) - \sum_{k=0}^{n} D_k \boldsymbol{e}\right) = \bar{g}_t(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ with

$$\bar{g}_t(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = \min_{i=1,\cdots,l,\, j=0,\cdots,n} F_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}), \tag{8}$$

where for $i = 1, \cdots, l$ and $j = 0, \cdots, n$

$$\begin{aligned} F_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = \min_{\bar{z}_j \leq y_j \leq \min\{\bar{z}_{j+1}, t_i\}, \bar{v}_{i-1} \leq t_i \leq \bar{v}_i} & \left[\sum_{k=0}^{i-1} \hat{s}_k \bar{v}_k + s_i t_i + \sum_{k=j+1}^{n} \hat{b}_k \bar{z}_k - b_j y_j + h(t_i - y_j)\right. \\ & \left. + \beta \bar{f}_{t+1}(\bar{z}_n, \cdots, \bar{z}_{j+1}, \overbrace{y_j, \cdots, y_j}^{j+1}, \overbrace{t_i, \cdots, t_i}^{i-1}, \bar{v}_i, \cdots, \bar{v}_{l-1}, \bar{v}_l)\right]. \end{aligned} \tag{9}$$

In equation (9), y_j is the negatively aggregated backorder quantity of classes $j,\cdots,n$, and t_i is the (net) inventory level at leadtime position i after inventory expediting and allocation. The condition $y_j \le t_i$ is required as, given we expedite up to leadtime position i and we fulfill demand up to class j, the total quantity of inventory that can be used to fulfill demands is bounded by the quantity of on-hand inventory plus the quantity of expedited inventory. Note that, if $\bar{z}_j > t_i$, then $\bar{z}_j \le y \le \min\{\bar{z}_{j+1}, t_i\}$ is infeasible and in this case we define $F_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}) \equiv \infty$. Finally, $\bar{f}_{T+l+1}(\boldsymbol{z},\boldsymbol{v}) \equiv 0$ for any $(\boldsymbol{z},\boldsymbol{v})$.

The optimal values of i and j in equation (8) indicate the highest leadtime position to be expedited from and the highest index of demand class to be fulfilled, respectively. Based on the optimal values of i and j, we then obtain the optimal expediting quantity and the optimal allocation quantity through equation (9). Note that, according to Corollary 1, once we decide to expedite from leadtime position i, we must have expedited all inventory at lower leadtime positions; similarly, if we decide to fulfill class j demand, we must have fulfilled all the demands of classes $0,\cdots,j-1$. This fact leads to the expression of $F_{ij,t}$ in equation (9).

Let $\Phi = (b_{n-1}-b_n,\cdots,b_0-b_1,0)$ and $\Theta = (s_2-s_1,\cdots,s_l-s_{l-1},0)$. We show in the following lemma the equivalence between the dynamic recursions in equations (7)-(8) and those in equations (1)-(2).

Lemma 3. *For $t=1,\cdots,T+l$, $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$ is the optimal cost from period t and onward given the state $(\boldsymbol{z},\boldsymbol{v})$ with the dynamic recursions in equations (7)-(8); moreover, $\bar{f}_t(\boldsymbol{z},\boldsymbol{v}) + \frac{\Phi'}{\beta}\boldsymbol{z}$ is nondecreasing in $z_1,\cdots,z_n$, and $\bar{f}_t(\boldsymbol{z},\boldsymbol{v}) + \frac{\Theta'}{\beta}\boldsymbol{v}$ is nondecreasing in $v_0,\cdots,v_{l-2}$.*

We derive the monotonicity of $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$ in Lemma 3 based on the sequential properties in Lemma 2.

Now define

$$
\begin{aligned}
(Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}), R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})) = \min \underset{\bar{z}_j \le y_j \le \min\{\bar{z}_{j+1},t_i\},\bar{v}_{i-1}\le t_i \le \bar{v}_i}{\arg\min} & \Bigg[\sum_{k=0}^{i-1} \hat{s}_k \bar{v}_k + s_i t_i + \sum_{k=j+1}^{n} \hat{b}_k \bar{z}_k - b_j y_j \\
& + h(t_i - y_j) + \beta \bar{f}_{t+1}(\bar{z}_n,\cdots,\bar{z}_{j+1},\overbrace{y_j,\cdots,y_j}^{j+1},\overbrace{t_i,\cdots,t_i}^{i-1},\bar{v}_i,\cdots,\bar{v}_{l-1},\bar{v}_l) \Bigg]
\end{aligned}
\tag{10}
$$

and

$$
(i_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}), j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})) = \max \underset{i=1,\cdots,l,\ j=0,\cdots,n}{\arg\min} F_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}). \tag{11}
$$

In equation (10), $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ is the negative value of the optimal aggregated backorder quantity of demand classes $j,\cdots,n$ while $R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ is the optimal net on-hand inventory level if we decide to fulfill demands from class 0 up to class j and expedite inventory from leadtime position 1 up to leadtime position i. In equation (11), $i_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ and $j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ are the largest indexes of leadtime position to expedite inventory and demand class to allocate inventory under the optimal policy, respectively. We also define

$$S_t^*(\boldsymbol{z},\boldsymbol{v}) = \min \mathop{\arg\min}_{v_l \geq v_{l-1}} \mathbb{E}\left[c(v_l - v_{l-1}) + \bar{g}_t\left(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, (\boldsymbol{v}, v_l) - \sum_{k=0}^{n} D_k \boldsymbol{e}\right)\right] \tag{12}$$

and let $q_t^*(\boldsymbol{z},\boldsymbol{v}) = S_t^*(\boldsymbol{z},\boldsymbol{v}) - v_{l-1}$ be the optimal ordering quantity.

THEOREM 1. *For $t = 1\cdots,T+l$, $i = 1,\cdots,l$, and $j = 0,\cdots,n$,*

(1) $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$, $\bar{g}_t(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ are $L^\natural$-convex.

(2) $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ and $R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ are nondecreasing in $(\bar{z}_n,\cdots,\bar{z}_j,\bar{v}_{i-1},\cdots,\bar{v}_l)$ and independent of $(\bar{z}_{j-1},\cdots,\bar{z}_0,\bar{v}_0,\cdots,\bar{v}_{i-2})$.

(3) $i_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ and $j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ are nondecreasing in $(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$.

(4) For the expediting decision, a state-dependent threshold policy is optimal: It is optimal to expedite up to leadtime position $i_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ and raise the current on-hand net inventory level up to the threshold $R_{i_t^* j_t^*,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ if it is below the threshold.

(5) For the allocation decision, a state-dependent rationing level policy is optimal: It is optimal to allocate inventory up to demand class $j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ and decrease the total backorder quantity of classes $j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}), j_t^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})+1,\cdots,n$ to the threshold $-Y_{i_t^* j_t^*,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ if it is above the threshold.

(6) For the ordering decision, a state-dependent base stock policy is optimal and the base stock level $S_t^*(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $(\boldsymbol{z},\boldsymbol{v})$. In addition, for any $\delta > 0$, $S_t^*(\boldsymbol{z},\boldsymbol{v}) \leq S_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \leq S_t^*(\boldsymbol{z},\boldsymbol{v})+\delta$, $q_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_{l-1}) \geq q_t^*(\boldsymbol{z},\boldsymbol{v}) - \delta$, and $q_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \leq q_t^*(\boldsymbol{z},\boldsymbol{v})$.

REMARK 2. As we have mentioned, the $L^\natural$-convexity still holds for the case with sufficiently large $s_1,\cdots,s_l$, i.e., the case with no expediting.

The property that $S_t^*(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $(\boldsymbol{z},\boldsymbol{v})$ can be explained as follows. Under the state-independent base stock policy, the impacts of backorders of different demand classes on the optimal ordering quantity are equal. However, in our case, lower-priority demand classes and the outstanding orders at lower leadtime positions may have less impact on the optimal ordering policy.

We also analyze how unit backordering costs affect the optimal ordering policy as follows. We describe the dependency of the optimal ordering policy on $\tilde{\boldsymbol{b}} = (-\hat{b}_0, -\hat{b}_1,\cdots,-\hat{b}_n)$. Let $\bar{f}_t(\tilde{\boldsymbol{b}},\boldsymbol{z},\boldsymbol{v})$ be the optimal value function and $q_t^*(\tilde{\boldsymbol{b}},\boldsymbol{z},\boldsymbol{v})$ be the optimal ordering quantity.

LEMMA 4. *For $t = 1,\cdots,T+l$,*

(1) $\bar{f}_t(\tilde{\boldsymbol{b}},\boldsymbol{z},\boldsymbol{v})$ is submodular in $\tilde{\boldsymbol{b}},\boldsymbol{z},\boldsymbol{v}$, and as a result, $q_t^*(\tilde{\boldsymbol{b}},\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $\tilde{\boldsymbol{b}}$.

(2) $q_t^*(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}) \le q_t^*(b_0, 0, \cdots, 0, \boldsymbol{z}, \boldsymbol{v})$.

Note that $q_t^*(b_0, 0, \cdots, 0, \boldsymbol{z}, \boldsymbol{v})$ is the optimal ordering quantity for an inventory system with only one demand class and leadtime l under the unit backordering cost b_0.

In the following theorem, we translate the structural properties in Theorem 1 with respect to the state $(\boldsymbol{z}, \boldsymbol{v})$ back to the original state $(\boldsymbol{w}, \boldsymbol{x})$, which allows us to conduct sensitivity analysis for the optimal expediting and allocation quantities. Recall that $(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}})$ is the system state after the ordering decision and demand realization. Let $q_t^*(\boldsymbol{w}, \boldsymbol{x})$ be the minimum optimal ordering quantity, $a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}})$ be the minimum optimal allocation quantity for demand class j, and $o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}})$ be the minimum optimal expediting quantity from leadtime position i.

Theorem 2. *For any $\delta > 0$,*

$$\begin{aligned}
&-\delta \le q_t^*(\boldsymbol{w}, \boldsymbol{x} + \delta \boldsymbol{e}_l) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \le \cdots \le q_t^*(\boldsymbol{w}, \boldsymbol{x} + \delta \boldsymbol{e}_2) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \le q_t^*(\boldsymbol{w}, \boldsymbol{x} + \delta \boldsymbol{e}_1) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \\
&= q_t^*(\boldsymbol{w} - \delta \boldsymbol{e}_1, \boldsymbol{x}) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \le q_t^*(\boldsymbol{w} - \delta \boldsymbol{e}_2, \boldsymbol{x}) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \le \cdots \le q_t^*(\boldsymbol{w} - \delta \boldsymbol{e}_{n+1}, \boldsymbol{x}) - q_t^*(\boldsymbol{w}, \boldsymbol{x}) \le 0,
\end{aligned}$$

and for $j = 0, \cdots, n$,

$$\begin{cases}
0 \ge a_{j,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_1, \bar{\boldsymbol{x}}) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) = \cdots = a_{j,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_j, \bar{\boldsymbol{x}}) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge -\delta, \\
\delta \ge a_{j,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_{j+1}, \bar{\boldsymbol{x}}) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge \cdots \ge a_{j,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_{n+1}, \bar{\boldsymbol{x}}) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge 0, \\
\delta \ge a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_1) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge \cdots \ge a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_{l+1}) - a_{j,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge 0.
\end{cases}$$

and for $i = 1, \cdots, l$,

$$\begin{cases}
0 \ge o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_1) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) = \cdots = o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_i) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge -\delta, \\
\delta \ge o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_{i+1}) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge \cdots \ge o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}} + \delta \boldsymbol{e}_{l+1}) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge 0, \\
\delta \ge o_{i,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_1, \bar{\boldsymbol{x}}) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge \cdots \ge o_{i,t}^*(\bar{\boldsymbol{w}} + \delta \boldsymbol{e}_{n+1}, \bar{\boldsymbol{x}}) - o_{i,t}^*(\bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) \ge 0.
\end{cases}$$

The results in Theorem 2 are explained as follows. First, the optimal ordering quantity is nonincreasing in the inventory level of any leadtime position. Increasing a unit of inventory at leadtime position i would lead to a smaller optimal ordering quantity than the counterpart at leadtime position $i-1$ for any $i = 2, \cdots, l$. Note that the inventory of each leadtime position i, $i = 1, \cdots, l$, is indeed the order placed in period $t - l + i$. We thus conclude that the optimal ordering quantity is more sensitive to the recently placed orders than to those placed earlier. Moreover, the optimal ordering quantity is nondecreasing in the backorder quantity of each demand class and it is more sensitive to the backorder quantities of higher-priority classes due to the higher service levels of those demand classes. Since class 0 backorders must be served as many as possible by the on-hand inventory, decreasing a unit of class 0 backorders has the same effect on the optimal ordering quantity as that of increasing a unit of on-hand inventory. Therefore, in general, the optimal ordering quantity is less sensitive to the backorder quantities than to the pipeline inventory levels.

Second, for the optimal allocation quantity to each demand class j, it is non-increasing in the backorder quantity of each demand class k for $k = 0, \cdots, j-1$, i.e., demand classes with higher priority than class j. The optimal allocation quantity to the demand class j is also nondecreasing in the backorder quantity of class k for $k = j, \cdots, n$, i.e., demand classes with lower priority than class j, and it is more sensitive to the backorder quantities of higher-priority classes. This property is consistent with our intuition that we shall serve more demands now if facing more backorders, especially when the backorders incur larger backordering costs. For the pipeline inventories, the optimal allocation quantity is nondecreasing in the inventory level of each leadtime position and is more sensitive to the inventory of lower leadtime positions, i.e., the sooner-to-arrival orders.

Finally, the optimal expediting quantity of leadtime position i is non-increasing in the inventory levels of leadtime positions $0, \cdots, i-1$ and has the same sensitivity to the inventory levels of these lower leadtime positions due to the sequential expediting property. It is also nondecreasing in the inventory levels of leadtime positions $i, \cdots, l$ and is more sensitive to the sooner-to-arrival orders. This property results from the facts that more expedited inventory is allowed when more pipeline inventory is available, and lower leadtime positions correspond to lower unit expediting costs. In addition, the optimal expediting quantity is nondecreasing in the backorder quantity of each demand class and it is more sensitive to the backorder quantities of higher-priority classes, which may be due to that more backorders of higher-priority classes would induce a larger expediting quantity in the current period.

The monotone properties in Theorem 2 not only enable us to understand how the optimal ordering, expediting, and allocation quantities change with the demand state and pipeline inventories, but also help us show how the optimal allocation policy can be simplified as follows.

Suppose that, after expediting, the inventory state is $\hat{\boldsymbol{x}} = (\hat{x}_0, \cdots, \hat{x}_l)$, where $\hat{x}_0 = \bar{x}_0 + \sum_{i=1}^{l} o_{i,t}$ and $\hat{x}_i = \bar{x}_i - o_{i,t}$ for $i = 1, \cdots, l$. Let $\hat{\boldsymbol{x}}_{-0} = (\hat{x}_1, \cdots, \hat{x}_l)$. We have the following result.

Theorem 3. *There exist rationing levels $r_{j,t}(\hat{\boldsymbol{x}}_{-0})$ for $j = 0, \cdots, n$, which are independent of the demand state $\bar{\boldsymbol{w}} = \boldsymbol{w} + \boldsymbol{D}$, such that the optimal allocation quantity of demand class j is given by $[\hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k - r_{j,t}(\hat{\boldsymbol{x}}_{-0})]^+ \wedge \bar{w}_j$. Moreover, $r_{0,t}(\hat{\boldsymbol{x}}_{-0}) = 0$.*

Theorem 3 is implied by the L$^\natural$-convexity of the value functions and the monotone properties in Theorem 2. Intuitively, the L$^\natural$-convexity implies that the optimal allocation quantity to demand class j is nondecreasing in the backorder quantity of class j and the on-hand inventory. Hence, if it is optimal to fulfill class j demand under any given state $(\boldsymbol{w}, \boldsymbol{x})$, then it must be optimal to fulfill at least δ amount of class j demand under state $(\boldsymbol{w} + \delta \boldsymbol{e}_{j+1}, \boldsymbol{x} + \delta \boldsymbol{e}_1)$. But after we fulfill δ

amount of class j demand, the state $(\boldsymbol{w}+\delta\boldsymbol{e}_{j+1},\boldsymbol{x}+\delta\boldsymbol{e}_1)$ becomes the state $(\boldsymbol{w},\boldsymbol{x})$. As a result, after optimal inventory allocation, the two states $(\boldsymbol{w}+\delta\boldsymbol{e}_{j+1},\boldsymbol{x}+\delta\boldsymbol{e}_1)$ and $(\boldsymbol{w},\boldsymbol{x})$ transit to the same state, which implies that the optimal rationing levels under these two states are the same. Coupled with the facts that $a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\geq\cdots\geq a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{n+1},\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\geq 0$ as in Theorem 2 and the optimal rationing levels are independent of x_0, we show that the optimal rationing levels are independent of the backorder quantities of different demand classes.

This result reduces the computational complexity of obtaining the optimal allocation policy as it only depends on the inventory levels of leadtime positions $1,\cdots,l$ after inventory expediting. The reason for the optimal rationing level of demand class j being independent of backorders of higher-priority demand classes is that we only fulfill class j demand when there are no backorders of higher-priority demand classes $0,\cdots,j-1$. The reason for the optimal rationing level of demand class j being independent of backorders of lower-priority demand classes is that demand class j has a higher priority over those demand classes.

Theorem 3 also implies the following proposition.

Proposition 1. *For a system with multiple demand classes and a zero replenishment leadtime, a state-independent rationing level policy is optimal for its allocation decision. The optimal allocation quantity of demand class j, $j=0,\ldots,n$, is $[\hat{x}_0-\sum_{k=0}^{j-1}\bar{w}_k-r_{j,t}]^+\wedge\bar{w}_j$, where $r_{j,t}$ is the rationing level for demand class j and $r_{0,t}=0$.*

Note that Topkis (1968) considers a system with multiple demand classes and only one chance of ordering before the planning horizon. He shows that for demand class j, $j=0,\cdots,n$, there exists a fixed rationing level $r_{j,t}$ and the optimal allocation quantity of demand class j is given by $[x-\sum_{k=0}^{j-1}(w_k+D_k)-r_{j,t}]^+\wedge(w_j+D_j)$. Proposition 1 generalizes the result in Topkis (1968) to the system with replenishments during the planning horizon. Our result is derived based on the L$^\natural$-convexity of the value functions.

5. The Optimality of Simple Policies

In the basic model, we partially characterize the structure of the optimal policy based on the properties of the L$^\natural$-convexity. However, the optimal policy in general is state-dependent and complex. In this section, we consider several special cases under which some simple *state-independent* policies turn out to be optimal. This in turn provides benchmarks and sheds light on how we can manage more general inventory systems with multiple demand classes. In the following, we focus on the systems with two or one demand class(es).

5.1. Systems with Two Demand Classes

5.1.1. Full Expediting Consider a system with two demand classes, namely classes 0 and 1, and a positive replenishment leadtime l. Different from the basic model, in this section, we assume that inventory can be expedited from any leadtime position and in addition an outside supplier with ample stock and a unit expediting cost s_{l+1}. Similarly, we impose the following assumption on the unit expediting costs to ensure the optimality of sequential expediting.

Assumption 2 **(Convex Expediting Cost Structure)**. *The unit expediting costs satisfy the following property:* $s_{i+1} - s_i \geq s_i - s_{i-1} \geq 0$ *for* $i = 1, \cdots, l$ $(s_0 \equiv 0)$.

For the ease of exposition, we refer to the outside supplier as leadtime position $l+1$. In addition to Assumption 2 on the convex expediting cost structure, we also assume that the unit backordering costs are sufficiently large such that $s_{l+1} < b_1$, i.e., the largest unit expediting cost is smaller than the smallest unit backordering cost. Then, the optimal inventory expediting policy is in fact intuitive: It is optimal to fulfill all the demands in each period by the on-hand inventory and the expedited inventory if needed, i.e., full expediting is optimal. Hence, in this case, our focus is to analyze the optimal ordering policy.

As it is optimal to fulfill any backordered demands, it is optimal to set $\boldsymbol{z} = 0$, i.e., there are no backorders after expediting. Hence, it suffices to use $\boldsymbol{v}$ as the system state in each period. Suppose that the ordering quantity is q and $v_l = v_{l-1} + q$. Then, the system state becomes $\bar{\boldsymbol{v}}$ after demand realization, where $\bar{\boldsymbol{v}} = (\bar{v}_0, \cdots, \bar{v}_l) = (v_0 - D_0 - D_1, \cdots, v_l - D_0 - D_1)$. After full expediting, the system state is $(\tilde{v}_0, \cdots, \tilde{v}_{l-1})$ such that $\tilde{v}_i = \bar{v}_{i+1} \vee 0$ for $i = 0, \cdots, l-1$. Then, the corresponding dynamic recursion is provided as follows:

$$\bar{f}_t(\boldsymbol{v}) = \min_{v_l \geq v_{l-1}} \mathbb{E}\left[c(v_l - v_{l-1}) + \bar{g}_t\left((\boldsymbol{v}, v_l) - (D_0 + D_1)\boldsymbol{e}\right)\right],$$

and

$$\bar{g}_t(\bar{\boldsymbol{v}}) = \sum_{i=1}^{l} s_i(\bar{v}_i - \bar{v}_i \vee 0 - \bar{v}_{i-1} + \bar{v}_{i-1} \vee 0) + s_{l+1}(\bar{v}_l \vee 0 - \bar{v}_l) + h(\bar{v}_0 \vee 0) + \beta \bar{f}_{t+1}(\tilde{\boldsymbol{v}}).$$

The terminal condition is $\bar{f}_{T+l+1}(\boldsymbol{v}) \equiv 0$ for any $\boldsymbol{v}$.

In the above formulas, all terms are intuitive except the term on expediting costs: We use $\bar{v}_i - \bar{v}_i \vee 0 - \bar{v}_{i-1} + \bar{v}_{i-1} \vee 0$ to denote the quantity of expedited inventory from leadtime position i. It can be explained as follows by three possible cases: (1) When the total inventory at leadtime positions $0, \cdots, i-1$ is sufficient to fully fulfill backorders, i.e., $\bar{v}_i \geq \bar{v}_{i-1} \geq 0$, the quantity of expedited inventory from position i is $\bar{v}_i - \bar{v}_i \vee 0 - \bar{v}_{i-1} + \bar{v}_{i-1} \vee 0 = 0$; (2) When the total inventory at leadtime positions $0,\cdots,i$ is sufficient to fully fulfill demands while the total inventory at leadtime positions $0,\cdots,i-1$ is insufficient, i.e., $\bar{v}_i \geq 0 > \bar{v}_{i-1}$, the quantity of expedited inventory from position i is $\bar{v}_i - \bar{v}_i \vee 0 - \bar{v}_{i-1} + \bar{v}_{i-1} \vee 0 = 0 - \bar{v}_{i-1}$; (3) When the total inventory at leadtime positions $0,\cdots,i$ is insufficient to fully fulfill demands, i.e., $\bar{v}_{i-1} \leq \bar{v}_i \leq 0$, the quantity of expedited inventory from position i is $\bar{v}_i - \bar{v}_i \vee 0 - \bar{v}_{i-1} + \bar{v}_{i-1} \vee 0 = \bar{v}_i - \bar{v}_{i-1}$.

For such a system, it turns out that a simple base stock policy is optimal.

PROPOSITION 2. *For the system with full expediting and $t = 1,\cdots,T+l+1$, the value function $\bar{f}_t(\boldsymbol{v})$ is decomposable and can be expressed as $\bar{f}_t(\boldsymbol{v}) = \sum_{i=0}^{l-1} \bar{f}_{i,t}(v_i)$, where each component function $\bar{f}_{i,t}(\cdot)$ is a single-variable convex function. A state-independent base stock policy is optimal for the ordering decision: In each period t, there exists a fixed base stock level S_t, which is given in (13), and it is optimal to order up to S_t if $v_{l-1} < S_t$ while order nothing otherwise.*

Given the component functions $\bar{f}_{i,t}(v_i)$'s for $i = 0,\cdots,l-1$, and let $D = D_0 + D_1$, the optimal base stock level is

$$S_t = \min \arg\min_{\zeta} \left[(c + s_l - s_{l+1})\zeta + \mathbb{E}[(s_{l+1} - s_l)((\zeta - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((\zeta - D) \vee 0)\right]. \tag{13}$$

For each period t, the base stock level S_t is a constant and independent of the system state. Due to Assumption 2, under the optimal policy, we still have the sequential expediting property and hence Lemma 3 remains valid under this setting. Note that the function $\beta \bar{f}_{l-1,t+1}(\zeta) + (s_{l+1} - s_l)\zeta$ is nondecreasing in ζ due to a similar argument as in Lemma 3. The monotone property of $\beta \bar{f}_{l-1,t+1}(\zeta) + (s_{l+1} - s_l)\zeta$ implies that, when $c + s_l \geq s_{l+1}$ (i.e., when c is large enough), it is optimal to order nothing in each period.

Next, we describe the algorithm on how to update the component functions. According to the component functions of $\bar{f}_{t+1}(\boldsymbol{v})$ and also the optimal base stock level S_t, we update the component functions of $\bar{f}_t(\boldsymbol{v})$ as follows:

$$\bar{f}_{0,t}(\zeta) = \mathbb{E}[-s_1(\zeta - D - (\zeta - D) \vee 0) + h((\zeta - D) \vee 0)], \tag{14}$$

$$\bar{f}_{i,t}(\zeta) = \mathbb{E}[(s_i - s_{i+1})(\zeta - D - (\zeta - D) \vee 0) + \beta \bar{f}_{i-1,t+1}((\zeta - D) \vee 0)], \quad i = 1,\cdots,l-2, \tag{15}$$

$$\begin{aligned}\bar{f}_{l-1,t}(\zeta) = {} & \mathbb{E}[(s_{l-1} - s_l)(\zeta - D - (\zeta - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((\zeta - D) \vee 0)] + c(\zeta \vee S_t) - c\zeta \\ & + \mathbb{E}[(s_l - s_{l-1})(\zeta \vee S_t - D - (\zeta \vee S_t - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((\zeta \vee S_t - D) \vee 0)].\end{aligned} \tag{16}$$

The details can be found in the proof of Proposition 2. By sequentially updating the component functions $\bar{f}_{i,t}(\zeta)$'s for $i = 0,\cdots,l-1$ from period $T+l+1$ to period 1, we can efficiently calculate the optimal base stock levels for all periods $t = 1,\cdots,T+l$ in a backward fashion. In this backward procedure, the initial component functions $\bar{f}_{i,T+l+1}(\zeta) = 0$ for $i = 0,\cdots,l-1$.

In addition, if demand D is continuous and i.i.d. in different periods, a myopic ordering policy can be optimal.

PROPOSITION 3. *For i.i.d. demands, if the initial inventory satisfies $v_{l-1} \le \tilde{S}$ in period 1, where $\tilde{S}$ is the myopic base stock level defined in equation (17), then a myopic base stock level policy with the base stock level $\tilde{S}$ is optimal in each period t, $t = 1, \cdots, T+l$.*

The component functions of $\bar{f}_t(\boldsymbol{v})$ defined in equations (14) and (15) imply that $\bar{f}_{0,t}(\zeta)$ is independent of t and can be explicitly written as in (14); $\bar{f}_{i,t}(\zeta)$ can then be recursively obtained for $i = 1, \cdots, l-2$ based on equation (15). The myopic base stock level $\tilde{S}$ is determined as follows

$$\tilde{S} = \min \arg\min_{\zeta} \left((c + s_l - s_{l+1})\zeta + (s_{l+1} - s_l)((\zeta - D) \vee 0) + \beta \mathbb{E}\left[\Phi_{t+1}((\zeta - D) \vee 0)\right] \right), \tag{17}$$

where $\Phi_t(\zeta) = \mathbb{E}\left[-c\zeta + (s_{l-1} - s_l)(\zeta - D - (\zeta - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((\zeta - D) \vee 0)\right]$ (noting that $\bar{f}_{l-2,t}$ can be explicitly obtained by equations (14)-(15) and is independent of t under the i.i.d. demands).

5.1.2. Partially Full Expediting: Full expediting may be too restrictive in some cases. In this section, we consider the scenario that the high-priority demand class requires full expediting while the low-priority demand class may not need it. Similar to the case with full expediting, we consider the system that allows inventory expediting from all leadtime positions and in addition an outside supplier (i.e., leadtime position $l+1$) with ample stock. We still impose Assumption 2 to ensure the optimality of sequential expediting. We also assume that the unit backordering cost of demand class 0 is sufficiently large so that $b_0 \ge s_{l+1}$, i.e., it is optimal to fulfill all class 0 demand in each period. Following Topkis (1968), there is no replenishment during the planning horizon in this setting. It is a reasonable assumption for many seasonal products as usually we do not have replenishment opportunities in those cases. Then, in this scenario, we only need to make expediting and allocation decisions for class 1 demand in each period.

Let $x_{i,1}$ be the initial inventory level of leadtime position i, $i = 0, \cdots, l+1$, before the planning horizon, which is a fixed value in our problem. We assume that $x_{l+1,1}$ is sufficiently large so that in the entire planning horizon all demands can be fully fulfilled by the inventory stocked at leadtime position $l+1$. We also denote by $u_{i,1} = -\sum_{k=i}^{l+1} x_{k,1}$ the negatively aggregated inventory level of leadtime position i at the beginning of period 1 for $i = 0, \cdots, l+1$ and define

$$u_{i,t} = \begin{cases} -\sum_{k=0}^{l+1} x_{k,1}, & i = 0, \\ -\sum_{k=\min\{l+1, i+t-1\}}^{l+1} x_{k,1}, & i = 1, \cdots, l, \\ -x_{l+1,1}, & i = l+1, \end{cases} \tag{18}$$

for $t=2,\cdots,T+l+1$. Clearly, $u_{i,t}$ for any $i=0,\cdots,l+1$ and $t=1,\cdots,T+l+1$ is independent of our decisions and demand realization. For the ease of exposition, we define $\boldsymbol{u}_t=(u_{0,t},\cdots,u_{l+1,t})$.

Let w be the backorder quantity of class 1 demand and v be the negative value of the total inventory of all the leadtime positions (which is referred to as the negative total inventory, or NTI) at the beginning of period t, and y be the NTI after the inventory expediting and allocation decisions in period t. Notice that we assume full fulfillment of class 0 demand. So there are no backorders of class 0 at the beginning of each period. We define $z=v+w$ as the negative of the inventory position (NIP), which is equal to NTI plus total backorder quantity, before demand realization and $\bar{z}=z+D_0+D_1$ as the NIP after demand realization.

We first show that the inventory state at the beginning of period t is $(v\vee u_{0,t},\cdots,v\vee u_{l+1,t})$. Note that, for period 1, it is clear that $v=u_{0,1}$ and the inventory state is $(v\vee u_{0,1},\cdots,v\vee u_{l+1,1})$. Suppose that the inventory state at the beginning of period t is $(v\vee u_{0,t},\cdots,v\vee u_{l+1,t})$, we then shall show that for period $t+1$ its inventory state has the same form. In the following, the analysis is performed in two steps. Specifically, we first show the inventory expediting and allocation decisions for class 0 demand and then analyze the decisions for class 1 demand.

In the first step, we fully fulfill class 0 demand after demand realization by the on-hand inventory and, if necessary, the expedited inventory. The NTI then becomes $\bar{v}=v+D_0$, and the inventory state of period t becomes $(\bar{v}\vee u_{0,t},\cdots,\bar{v}\vee u_{l+1,t})$ due to the sequential expediting property. Accordingly, the total expediting cost is

$$\sum_{i=1}^{l} s_i\left[(\bar{v}\vee u_{i,t})\wedge(v\vee u_{i+1,t})-v\vee u_{i,t}\right]+s_{l+1}\left(\bar{v}\vee u_{l+1,t}-v\vee u_{l+1,t}\right), \tag{19}$$

or equivalently

$$-s_1(v\vee u_{1,t})-\sum_{i=2}^{l+1}(s_i-s_{i-1})(v\vee u_{i,t})+s_1(\bar{v}\vee u_{1,t})+\sum_{i=2}^{l+1}(s_i-s_{i-1})(\bar{v}\vee u_{i,t}). \tag{20}$$

We explain the derivation of the expediting cost in Appendix A.

In the second step, we consider the optimal expediting and allocation decisions for class 1 demand. Recall that y is the NTI after inventory expediting and allocation at period t. Then, we must have allocated $y-\bar{v}$ units of inventory to fulfill class 1 demand. The inventory state is thus $(y\vee u_{0,t},y\vee u_{1,t},\cdots,y\vee u_{l+1,t})$ after inventory expediting and allocation. Note that $y\geq\bar{v}$ must hold and $y-\bar{v}\leq w+D_1$ as we at most fulfill $w+D_1$ units of class 1 demand at each period. Similar to the expression in (20), the expediting cost for fulfilling class 1 demand is

$$-s_1(\bar{v}\vee u_{1,t})-\sum_{i=2}^{l+1}(s_i-s_{i-1})(\bar{v}\vee u_{i,t})+s_1(y\vee u_{1,t})+\sum_{i=2}^{l+1}(s_i-s_{i-1})(y\vee u_{i,t}).$$

As the inventory at leadtime position i, $i = 1, \cdots, l$, moves one position forward in each period, the inventory state at the beginning of period $t+1$ shall be $(y \vee u_{0,t}, y \vee u_{2,t}, \cdots, y \vee u_{l+1,t}, y \vee u_{l+1,t})$ or equivalently $(y \vee u_{0,t+1}, y \vee u_{1,t+1}, \cdots, y \vee u_{l,t+1}, y \vee u_{l+1,t+1})$ according to (18).

Based on the above analysis, we have shown that the inventory state at the beginning of period t is $(v \vee u_{0,t}, \cdots, v \vee u_{l+1,t})$ for $t = 1, \cdots, T+l+1$. Then, the dynamic recursion of this problem is given as follows:

$$\bar{g}_t(z, v \vee \boldsymbol{u}_t) = \mathbb{E}\left[-s_1(v \vee u_{1,t}) - \sum_{i=2}^{l+1}(s_i - s_{i-1})(v \vee u_{i,t}) + \hat{g}_t(z + D_0 + D_1, (v + D_0) \vee \boldsymbol{u}_t)\right]$$

where $\hat{g}_t(z + D_0 + D_1, (v + D_0) \vee \boldsymbol{u}_t) = \hat{g}_t(\bar{z}, \bar{v} \vee \boldsymbol{u}_t)$ and

$$\hat{g}_t(\bar{z}, \bar{v} \vee \boldsymbol{u}_t) = \min_{\bar{v} \le y \le \bar{z}} \left[s_1(y \vee u_{1,t}) + \sum_{i=2}^{l+1}(s_i - s_{i-1})(y \vee u_{i,t}) + \beta \bar{g}_{t+1}(\bar{z}, y \vee \boldsymbol{u}_{t+1})\right]. \tag{21}$$

The terminal condition is $\bar{g}_t(z, v \vee \boldsymbol{u}_{T+l+1}) \equiv 0$ for any (z, v).

Proposition 4. *For the system with partially full expediting, in each period t, $t = 1, \cdots, T+l+1$, the function $\bar{g}_t(z, v \vee \boldsymbol{u}_t)$ is decomposable and can be expressed as $\bar{g}_t(z, v \vee \boldsymbol{u}_t) = \tilde{g}_t(z) + \bar{g}_{0,t}(v) + \sum_{i=1}^{l+1} \bar{g}_{i,t}(v \vee u_{i,t})$, where $u_{i,t}$'s are fixed values. The optimal expediting and allocation policy for class 1 demand can be described by a state-independent rationing level policy: There exists a fixed rationing level R_t and it is optimal to use $(v + D_0) \vee (R_t \wedge (z + D_0 + D_1))$ units of inventory to fulfill class 1 demand, where the inventory is sequentially expedited from leadtime positions $0, 1, \cdots, l+1$ if needed.*

To calculate R_t in each period t, we shall adopt a backward procedure starting from the terminal period $T+l+1$ with $\tilde{g}_{T+l+1}(\zeta) \equiv 0$ and $\bar{g}_{i,T+l+1}(\zeta) \equiv 0$ for $i = 0, \cdots, l+1$. Suppose that, in period $t+1$, the function $\bar{g}_{t+1}(z, v \vee \boldsymbol{u}_{t+1})$ is decomposable and $\bar{g}_{t+1}(z, v \vee \boldsymbol{u}_{t+1}) = \tilde{g}_{t+1}(z) + \bar{g}_{0,t+1}(v) + \sum_{i=1}^{l+1} \bar{g}_{i,t+1}(v \vee u_{i,t+1})$. Then, in period t, $t = 1, \cdots, T+l$,

$$R_t = \min \arg\min_{y} \left[s_1(y \vee u_{1,t}) + \sum_{i=2}^{l+1}(s_i - s_{i-1})(y \vee u_{i,t}) + \beta \bar{g}_{0,t+1}(y) + \beta \sum_{i=1}^{l+1} \bar{g}_{i,t+1}(y \vee u_{i,t+1})\right].$$

The component function of $\bar{g}_t(z, v \vee \boldsymbol{u}_t)$ is updated as follows:

$$\begin{cases} \tilde{g}_t(\zeta) = \mathbb{E}\left[\hat{G}_{1,t}(\zeta + D_0 + D_1)\right], \\ \bar{g}_{0,t}(\zeta) = \mathbb{E}\left[\hat{G}_{2,t}(\zeta + D_0)\right], \\ \bar{g}_{1,t}(\zeta) = -s_1 \zeta, \\ \bar{g}_{i,t}(\zeta) = (s_{i-1} - s_i)\zeta, \qquad\qquad i = 1, \cdots, l+1, \end{cases}$$

where

$$\begin{cases} \hat{G}_{1,t}(\zeta) = s_1(u_{1,t} \vee \zeta - u_{1,t} \vee R_t \vee \zeta) + \sum_{i=2}^{l+1}(s_i - s_{i-1})(u_{i,t} \vee \zeta - u_{i,t} \vee R_t \vee \zeta) + \beta\tilde{g}_{t+1}(\zeta) \\ \qquad +\beta\bar{g}_{0,t+1}(\zeta) + \beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t} \vee \zeta) + \beta\bar{g}_{l+1,t+1}(u_{l+1,t} \vee \zeta) \\ \qquad -\beta\bar{g}_{0,t+1}(R_t \vee \zeta) - \beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t} \vee R_t \vee \zeta) - \beta\bar{g}_{l+1,t+1}(u_{l+1,t} \vee R_t \vee \zeta), \\ \hat{G}_{2,t}(\zeta) = s_1(u_{1,t} \vee R_t \vee \zeta) + \sum_{i=2}^{l+1}(s_i - s_{i-1})(u_{i,t} \vee R_t \vee \zeta) \\ \qquad +\beta\bar{g}_{0,t+1}(R_t \vee \zeta) + \beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t} \vee R_t \vee \zeta) + \beta\bar{g}_{l+1,t+1}(u_{l+1,t} \vee R_t \vee \zeta). \end{cases}$$

By sequentially calculating R_t and updating the component functions of $\bar{g}_t(z, v \vee \boldsymbol{u}_t)$ from period $T+l+1$ to period 1, we can efficiently obtain the rationing level policy for the entire planning horizon.

Notice that although Topkis (1968) shows the optimality of a fixed rationing level policy for the inventory problem with priority demands, he does not show the decomposition of the value functions. The decomposition of the value functions in Proposition 4 allows us to obtain the optimal rationing levels efficiently.

5.2. Systems with One Demand Class

To isolate the effect of expediting on the optimal ordering policy, in this section, we consider the system with only one demand class. As the same with the basic model, inventory expediting is allowed at leadtime positions $1, \cdots, l$ only. We show in the following that a base stock policy and a *calibrated rationing level policy* are optimal for the ordering and expediting decisions, respectively, in each period.

For a system with one demand class, the dynamic recursion can be simplified to

$$\bar{f}_t(\boldsymbol{v}) = \min_{v_l \geq v_{l-1}} \mathbb{E}[c(v_l - v_{l-1}) + \bar{g}_t((\boldsymbol{v}, v_l) - D\boldsymbol{e})],$$

where, with $\bar{\boldsymbol{v}} = (\boldsymbol{v}, v_l) - D\boldsymbol{e}$ and $\bar{\boldsymbol{v}}_+ = (\bar{v}_1, \cdots, \bar{v}_l)$,

$$\bar{g}_t(\bar{\boldsymbol{v}}) = \min_{\bar{v}_0 \leq y \leq \bar{v}_l} \left[h(y \vee 0) + b_0(y \vee 0 - y) + \sum_{i=1}^{l} s_i(\bar{v}_i - y \vee \bar{v}_i - \bar{v}_{i-1} + y \vee \bar{v}_{i-1}) + \beta\bar{f}_{t+1}(y \vee \bar{\boldsymbol{v}}_+) \right].$$

The terminal condition is $\bar{f}_{T+l+1}(\boldsymbol{v}) \equiv 0$ for any $\boldsymbol{v}$.

We explain the expression of $\bar{g}_t(\bar{\boldsymbol{v}})$ as follows. Note that y is the net on-hand inventory level after expediting. Consider a scenario that $\bar{v}_{i-1} \leq y \leq \bar{v}_i$, i.e., we expedite inventory up to leadtime position i, $i = 1, \cdots, l$. Then, at the beginning of the next period, the net inventory levels of leadtime positions $0, \cdots, i-2$ are all equal to y, while the net inventory levels of leadtime positions $i-1, \cdots, l-1$ are $\bar{v}_i, \cdots, \bar{v}_l$. For the holding cost, if $y > 0$ then $y \vee 0 = y$, and there is a holding cost hy; otherwise, $y \leq 0$ and $y \vee 0 = 0$, and there is no holding cost. For the backordering cost, $y < 0$ means that $y \vee 0 - y = -y$ and the backordering cost is $-b_0 y$ while $y \geq 0$ means that $y \vee 0 - y = 0$

and there is no backordering cost. The expediting cost can be explained based on the same logic as in Section 5.1.

We then show the following result for the system with one demand class.

PROPOSITION 5. *In the system with one demand class, the value functions $\bar{f}_t(\boldsymbol{v})$ and $\bar{g}_t(\bar{\boldsymbol{v}})$ are convex for $t=1,\cdots,T+l+1$. Moreover, the function $\bar{f}_t(\boldsymbol{v})$ is decomposable and can be expressed as $\bar{f}_t(\boldsymbol{v})=\sum_{i=0}^{l-1}\bar{f}_{i,t}(v_i)$, where $\bar{f}_{i,t}(\zeta)$ for $i=0,\cdots,l-1$ are single-variable convex functions. The optimal ordering policy is a base stock policy: There exists a fixed base stock level S_t such that it is optimal to order up to the base stock level when $v_{l-1}<S_t$ and order nothing otherwise. The optimal expediting policy is the following calibrated rationing level policy: There exists a set of fixed rationing levels $r_{1,t},\cdots,r_{l,t}$ such that $r_{1,t}\geq\cdots\geq r_{l,t}$; it is optimal to sequentially expedite inventory from leadtime positions $1,\cdots,l$ so that the inventory state at the end of period t becomes $(R_t\vee\bar{v}_0,\cdots,R_t\vee\bar{v}_l)$, where R_t is the calibrated rationing level defined as $R_t=\sum_{i=1}^{l}(\bar{v}_{i-1}\vee r_{i,t})\wedge\bar{v}_i-\sum_{i=1}^{l-1}\bar{v}_i$.*

We show how to update the component functions $\bar{g}_{i,t}(\cdot)$ for $i=0,\cdots,l$ and $\bar{f}_{i,t}(\cdot)$ for $i=0,\cdots,l-1$ based on the component functions $\bar{f}_{i,t+1}(\cdot)$'s in the proof of this proposition. The logic is the same as shown in the proof of Proposition 2. Again, with a backward procedure, we can obtain the optimal policy efficiently.

For this system, a base stock policy is optimal for the ordering decision and the base stock level is

$$S_t=\min\arg\min_{\zeta}\mathbb{E}\left[c\zeta+\bar{g}_{l,t}(\zeta-D_t)\right],$$

which is independent of the system state in period t. By updating the component function $\bar{g}_{l,t}(\cdot)$ in the backward fashion, it is easy to calculate S_t's sequentially for periods $t=T+l,T+l-1,\cdots,1$.

We then explain the calibrated rationing level policy below. In each period, we obtain a set of constants $\{r_{1,t},\cdots,r_{l,t}\}$, where

$$r_{i,t}=\min\arg\min_{\zeta}\left[(h+b_0)(\zeta\vee 0)+(s_1-b_0)\zeta-\sum_{k=1}^{i-1}(s_k-s_{k+1})\zeta+\beta\sum_{k=1}^{i-1}\bar{f}_{k-1,t+1}(\zeta)\right]$$

for $i=1,\cdots,l$.

We refer to $r_{i,t}$'s as the fixed rationing levels, which are independent of system state and can be efficiently calculated in a backward fashion. Essentially, $r_{i,t}$ is the global minimizer of the function $(h+b_0)(y\vee 0)-b_0y+\sum_{i=1}^{l}s_i(\bar{v}_i-y\vee\bar{v}_i-\bar{v}_{i-1}+y\vee\bar{v}_{i-1})+\beta\bar{f}_{t+1}(y\vee\bar{v}_1,\cdots,y\vee\bar{v}_l)$ when $\bar{v}_{i-1}\leq y\leq\bar{v}_i$. Hence, if we decide to expedite up to leadtime position i, $i\in\{1,\cdots,l\}$, in period t, it is optimal to let alone the net on-hand inventory level after expediting be the fixed rationing level $r_{i,t}$ when $\bar{v}_{i-1} \le r_{i,t} \le \bar{v}_i$. However, when $r_{i,t} < \bar{v}_{i-1}$ ($r_{i,t} > \bar{v}_i$), we expedite inventory up to leadtime position k for some k such that $k < i$ ($k > i$) under the rationing level $r_{k,t}$. This suggests that a traditional rationing level policy may not be optimal. Instead, the optimal inventory expediting decision depends on both the fixed rationing levels and the inventory state $\bar{\boldsymbol{v}}$. We hence refer to the rationing policy as a calibrated rationing level policy, where the rationing level policy is calibrated by the inventory state.

Though the calibrated rationing level policy depends on both the fixed rationing levels and the inventory state, due to the sequential properties $r_{1,t} \ge \cdots \ge r_{l,t}$ and $\bar{v}_0 \le \cdots \le \bar{v}_l$, it is clear that

$$\begin{aligned} R_t &= \sum_{i=1}^{l} \mathbb{I}_{\{\bar{v}_{i-1} \le r_{i,t} < \bar{v}_i\}} r_{i,t} + \sum_{i=1}^{l-1} \mathbb{I}_{\{r_{i+1,t} < \bar{v}_i \le r_{i,t}\}} \bar{v}_i + \mathbb{I}_{\{r_{1,t} < \bar{v}_0\}} \bar{v}_0 + \mathbb{I}_{\{r_{l,t} \ge \bar{v}_l\}} \bar{v}_l \\ &= \sum_{i=1}^{l} (\bar{v}_{i-1} \vee r_{i,t}) \wedge \bar{v}_i - \sum_{i=1}^{l-1} \bar{v}_i. \end{aligned} \tag{22}$$

The equivalence between the above two expressions of R_t has been shown in the proof of Proposition 5. We then explain why we define the calibrated rationing level as in equation (22) below. If $\bar{v}_{i-1} \le r_{i,t} \le \bar{v}_i$ for some i, then $r_{k,t} \ge \bar{v}_k$ for $k = 1, \cdots, i-1$ and $r_{k,t} \le \bar{v}_{k-1}$ for $k = i+1, \cdots, l$. In this case, $R_t = r_{i,t}$. That is, it is optimal to expedite inventory up to leadtime position i and the post-expediting state is $(\underbrace{r_{i,t}, \cdots, r_{i,t}}_{i}, \bar{v}_i, \cdots, \bar{v}_l)$. Similarly, if $r_{i+1,t} \le \bar{v}_i \le r_{i,t}$ for some i, then $r_{k,t} \ge \bar{v}_k$ for $k = 1, \cdots, i$ and $r_{k,t} \le \bar{v}_{k-1}$ for $k = i+1, \cdots, l$. In this case, it is optimal to set $R_t = \bar{v}_i$ so that the inventories at leadtime positions $1, \cdots, i$ are fully expedited and the post-expediting state is $(\underbrace{\bar{v}_i, \cdots, \bar{v}_i}_{i+1}, \bar{v}_{i+1}, \cdots, \bar{v}_l)$.

In summary, for the system with one demand class, a base stock policy is optimal for the ordering decisions and a calibrated rationing level policy is optimal for the inventory expediting decisions. The calibrated rationing level R_t depends on both the fixed rationing levels $r_{i,t}$ for $i = 1, \cdots, l$ and the state $\bar{\boldsymbol{v}}$ after demand realization. Essentially, to implement the optimal policy, we only need to calculate the fixed base stock level S_t and the fixed rationing levels $r_{1,t}, \cdots, r_{l,t}$ for each period $t = 1, \cdots, T+l$. After obtaining these constants, we can directly implement the optimal policy based on the system state $\boldsymbol{v}$ and $\bar{\boldsymbol{v}}$. The constants S_t and $r_{i,t}$'s can be efficiently obtained through the updating of component functions in a backward fashion.

If demands are continuous and i.i.d. in different periods, the rationing level $r_{i,t}$ is independent of t and denoted by r_i for $i = 1, \cdots, l$. Moreover, similar to Section 5.1.1, we can obtain the closed expressions of component functions $\bar{f}_{i,t}(v_i)$ for $i = 0, \cdots, l-2$. We then have the following result.

Proposition 6. *Under i.i.d. demands for each period, a myopic ordering policy is optimal with the myopic base stock level $\hat{S}$ defined in (23) if $v_{l-1,1} \le \hat{S}$ in period 1.*

The myopic base stock level $\hat{S}$ is determined as

$$\begin{aligned}\hat{S} = \min\arg\min_{\zeta}\Bigg\{ & c\zeta + \mathbb{E}\left[(h+b_0)((\zeta-D)\vee 0 - r_l \vee(\zeta-D)\vee 0) + (s_{l+1}-b_0)(\zeta - D - r_l\vee(\zeta-D))\right] \\ & + \beta\mathbb{E}\left[\sum_{k=1}^{l-1}\bar{f}_{k-1,t+1}(\zeta-D) - \sum_{k=1}^{l-1}\bar{f}_{k-1,t+1}(r_l\vee(\zeta-D))\right] + \beta\mathbb{E}\left[U_{t+1}(\zeta-D)\right]\Bigg\},\end{aligned} \tag{23}$$

where $U_t(\zeta) = -c\zeta + \mathbb{E}\left[\bar{g}_{l-1,t}(\zeta-D)\right]$ and

$$\begin{aligned}\bar{g}_{l-1,t}(\zeta) = & (h+b_0)(r_l\vee\zeta\vee 0 - r_{l-1}\vee\zeta\vee 0) + (s_l - b_0)(r_l\vee\zeta - r_{l-1}\vee\zeta) + (s_{l-1}-s_l)(\zeta - r_1\vee\zeta) \\ & + \sum_{k=2}^{l-1}(s_{l-1}-s_l)(r_{k-1}\vee\zeta - r_k\vee\zeta) + \beta\sum_{k=1}^{l-1}\bar{f}_{k-1,t+1}(r_l\vee\zeta) - \beta\sum_{k=1}^{l-2}\bar{f}_{k-1,t+1}(r_{l-1}\vee\zeta).\end{aligned}$$

6. Numerical Studies and Operational Insights

In this section, we investigate the properties of the optimal policy and the performances of three proposed heuristics numerically. We then derive operational insights on how to manage inventory systems with multiple demand classes and expediting.

We focus on systems with two demand classes and a one-period leadtime. Consistent with the notation in Section 4, let (w_0, w_1) and (D_0, D_1) be the backorders and the realized demands, respectively. We also use (x, q) to represent the on-hand inventory and the ordering quantity in each period.

6.1. Illustrations of the Optimal Policy

We first numerically illustrate the optimal policy of our problem in period 1 under the following parameter setting: $T = 40$, $b_0 = 0.8$, $b_1 = 0.4$, $h = 0.3$, $c = 0.5$, $s_1 = 0.5$ and $\beta = 0.95$. We assume that demands of both classes follow the same truncated normal distribution $N(\mu, \sigma)$ with $\mu = 2$ and $\sigma/\mu = 0.5$. Table 1 presents how the optimal ordering quantity $q^*(w_0, w_1, x)$ changes with respect to different system states. The numerical result is consistent with the sensitivity analysis on $q^*(\boldsymbol{w}, \boldsymbol{x})$ in Theorem 2 as, e.g., $q^*(2,4,6) = q^*(1,4,5)$ and $q^*(2,4,6) - q^*(2,4,5) = -1$. Table 2 shows the optimal allocation policy under different system states. The result coincides with the sensitivity analysis on $a_j^*(\bar{w}_0, \bar{w}_1, x, q)$ in Theorem 2. For example, $a_0^*(2,5,5,7) - a_0^*(2,4,5,7) = 0$, $a_1^*(3,4,5,7) - a_1^*(2,4,5,7) = -1$ and $a_1^*(2,4,4,10) > a_1^*(2,4,3,11)$. In Table 3, we present the total expediting quantity of two demand classes under the optimal expediting policy. Due to the sequential allocation property under the optimal allocation policy, the expedited inventory shall be used to fulfill class 0 demand first and fulfill class 1 demand only if class 0 demand is fully fulfilled. As in our setting $s_1 > b_1$ and the leadtime is only one period, intuitively we shall not expedite inventory for serving class 1 demand. Hence, in Table 3, the total expediting quantity is equal to the expediting quantity for class 0. Then, $o_0^*(\bar{w}_0, \bar{w}_1, x, q)$ is consistent with the inequalities in Theorem 2 as, for example, $o_0^*(6,6,5,10) - o_0^*(4,6,5,10) > o_0^*(4,8,5,10) - o_0^*(4,6,5,10)$.

Table 1 The optimal order quantity $q^*(w_0, w_1, x)$.

Case (1) $((w_1, x) = (4,5))$		Case (2) $((w_0, x) = (2,5))$		Case (3) $((w_0, w_1) = (2,4))$	
w_0	$q^*(w_0, w_1, x)$	w_1	$q^*(w_0, w_1, x)$	x	$q^*(w_0, w_1, x)$
0	7	0	5	0	14
1	8	1	6	1	13
2	9	2	7	2	12
3	10	3	8	3	11
4	11	4	9	4	10
5	12	5	10	5	9
6	13	6	11	6	8
7	14	7	12	7	7
8	15	8	13	8	6
9	16	9	14	9	5

Table 2 The optimal allocation quantity $a_j^*(\bar{\boldsymbol{w}}, x, q)$ for $j = 0, 1$.

Case (1) $((\bar{w}_1, x, q) = (4,5,7))$			Case (2) $((\bar{w}_0, x, q) = (2,5,7))$			Case (3) $((\bar{w}_0, \bar{w}_1) = (2,4))$			
$\bar{w}_0$	$a_0^*(\bar{\boldsymbol{w}}, x, q)$	$a_1^*(\bar{\boldsymbol{w}}, x, q)$	$\bar{w}_1$	$a_0^*(\bar{\boldsymbol{w}}, x, q)$	$a_1^*(\bar{\boldsymbol{w}}, x, q)$	x	q^*	$a_0^*(\bar{\boldsymbol{w}}, x, q)$	$a_1^*(\bar{\boldsymbol{w}}, x, q)$
0	0	4	2	2	2	2	12	2	0
1	1	4	3	2	3	3	11	2	1
2	2	3	4	2	3	4	10	2	2
3	3	2	5	2	3	5	9	2	3
4	4	1	6	2	3	6	8	2	4
5	5	0	7	2	3	7	7	2	4

Table 3 The total expediting quantity $o^*(\bar{\boldsymbol{w}}, x, q)$ for two demand classes.

Case (1) $((\bar{w}_1, x, q) = (6,4,10))$		Case (2) $((\bar{w}_0, x, q) = (4,4,10))$		Case (3) $((\bar{w}_0, \bar{w}_1) = (4,6))$		
$\bar{w}_0$	$o^*(\bar{\boldsymbol{w}}, x, q)$	$\bar{w}_1$	$o^*(\bar{\boldsymbol{w}}, x, q)$	x	q^*	$o^*(\bar{\boldsymbol{w}}, x, q)$
3	0	4	0	0	14	4
4	0	5	0	1	13	3
5	0	6	0	2	12	2
6	1	7	0	3	11	1
7	2	8	0	4	10	0

6.2. Simple Heuristics

As it is computationally extensive to obtain the optimal policy, to simplify the computation, we propose three heuristics for our problem based on our analytical results in Section 5. We consider the system under which inventory can be expedited from an outside supplier with ample stock and a unit expediting cost s_2 $(s_2 - s_1 \geq s_1)$. We compare the performances of the three proposed heuristics against the optimal policy through extensive numerical studies.

Under the first heuristic, we expedite as much inventory as possible to fulfill all the demands of both classes. We refer to it as the *full expediting policy.* Based on the result in Section 5.1.1, we use a myopic base stock policy with the myopic base stock level s in this heuristic for the ordering decision and, in each period, we order $\max(0, s - (x - w_0 - w_1))$ units of inventory. Demands are fulfilled by the on-hand inventory first and then the inventory sequentially expedited from leadtime positions if necessary.

The second heuristic does not allow any inventory expediting. We still adopt a myopic base stock policy and simply fulfill high-priority demand first and then low-priority demand with the on-hand inventory as much as possible. We refer to this heuristic as the *no expediting policy.*

Under the third heuristic, we adopt a myopic base stock policy for the ordering decision and, motivated by the result in Section 5.1.2, a fixed rationing level policy with a nonnegative and fixed threshold r_i for the inventory allocation and expediting decisions of demand class i, $i = 0, 1$. Specifically, we fulfill high-priority demand as much as possible with the on-hand inventory and, if the on-hand inventory is not sufficient, we sequentially expedite $\max(0, w_0 + D_0 - x - r_0)$ units of inventory from leadtime positions. For low-priority demand, we first fulfill the demand with the on-hand inventory as much as possible and then sequentially expedite $\max(0, w_1 + D_1 - \min(\max(0, x - w_0 - D_0), w_1 + D_1) - r_1)$ units of inventory from leadtime positions. We refer to this heuristic as the *static policy.* The static policy reduces to the full expediting policy when $r_i = 0$ for $i = 0, 1$, and the no expediting policy when $r_i \to +\infty$ for $i = 0, 1$.

For the three proposed heuristics, we obtain the best parameters of s, r_0 and r_1 by using exhaustive grid search (the myopic base stock level is chosen to minimize the expected myopic inventory cost as in Section 5).

Let $C(\mathcal{P})$ be the total expected cost under the policy $\mathcal{P}$, and define the performance gap between a heuristic and the optimal policy as follows:

$$\xi(\mathcal{P}) = \frac{C(\mathcal{P}) - C(\text{Opt})}{C(\text{Opt})} \times 100\%,$$

where $\mathcal{P} \in \{\mathrm{F}, \mathrm{N}, \mathrm{S}\}$ ("F", "N" and "S" refer to the full expediting policy, the no expediting policy and the static policy, respectively). The system parameters are set as follows. We consider the planning horizon as $T = 40$ with a discount factor $\beta = 0.95$, and set $b_0 = 0.8$, $b_1 = 0.4$, $h = 0.3$, $c = 0.5$, $s_1 = 0.5$, $s_2 = 1$, $\mu = 2$ and $\sigma = 1$. We then investigate the performance gaps of the three proposed heuristics by varying one of the parameters while keeping the other parameters fixed. Specifically, we consider the unit backordering costs $b_0 \in \{0.6, 0.7, 0.8, 0.9, 1.0\}$, $b_1 \in \{0.2, 0.3, 0.4, 0.6, 0.8\}$, the unit holding cost $h \in \{0.1, 0.2, 0.3, 0.4, 0.5\}$, the unit expediting costs $s_1 \in \{0.2, 0.3, 0.5, 0.6, 0.7\}$, $s_2 \in \{0.6, 0.8, 1.0, 1.2, 1.4\}$, the unit ordering cost $c \in \{0.2, 0.3, 0.5, 0.6, 0.7\}$, and $\frac{\sigma}{\mu} \in \{0.1, 0.2, 0.3, 0.5, 0.7\}$ for the demand distribution. By setting $(x, w_0, w_1) = (5, 0, 0)$, we present the numerical results in Table 4.

Table 4 **Performance gaps (%) of three heuristics under various parameter settings.**

b_0	ξ(F)	ξ(N)	ξ(S)	b_1	ξ(F)	ξ(N)	ξ(S)	h	ξ(F)	ξ(N)	ξ(S)
0.6	3.82	0.88	0.82	0.2	11.61	1.53	0.96	0.1	2.62	0.99	0.97
0.7	3.8	0.94	0.8	0.3	7.11	1.11	0.9	0.2	2.53	0.93	0.99
0.8	3.78	0.99	0.78	0.4	3.78	0.99	0.78	0.3	3.78	0.99	0.78
0.9	3.77	1.06	0.77	0.6	0.63	3.8	0.63	0.4	3.70	0.99	0.79
1	3.76	1.12	0.76	0.8	0.54	5.36	0.54	0.5	3.62	1.69	0.8
s_1	ξ(F)	ξ(N)	ξ(S)	s_2	ξ(F)	ξ(N)	ξ(S)	c	ξ(F)	ξ(N)	ξ(S)
0.2	0.57	8.76	0.57	0.6	0.13	12.59	0.13	0.2	5.27	0.89	0.49
0.3	0.74	4.25	0.74	0.8	3.78	1	0.78	0.3	4.63	0.91	0.6
0.5	3.78	0.99	0.78	1	3.78	0.99	0.78	0.5	3.78	0.99	0.78
0.6	5.51	0.92	0.79	1.2	3.79	0.99	0.78	0.6	3.46	1.05	0.91
0.7	7.02	0.84	0.81	1.4	3.8	0.99	0.78	0.7	3.25	1.12	0.99
σ/μ	ξ(F)	ξ(N)	ξ(S)								
0.1	0.95	0.84	0.84								
0.2	2.19	0.82	0.82								
0.3	2.88	0.82	0.81								
0.5	3.78	0.99	0.78								
0.7	3.7	1.4	0.83								

In Table 4, the performance of the static policy is the best among the three heuristics, which is as expected since the full and no expediting policies are its two special cases. The static policy is recommended under the following scenarios: (1) b_0 is large but b_1 is small or moderate, (2) h is large, (3) the coefficient of variation (CoV) σ/μ is large. Under the scenario (1), intuitively a large difference $b_0 - b_1$ implies that we shall fulfill more high-priority demand but less low-priority demand with expediting. The two simple policies – the full expediting policy and the no expediting policy – thus perform significantly worse than the static policy. Under the scenario (2), a larger value of h usually leads to a smaller order-up-to level s and hence the fulfillment of demands relies more on expediting. However, inappropriate expediting would lead to the increase of the system cost. Under the scenario (3), both the full expediting policy and the no expediting policy would incur some unnecessary costs and hence appropriate expediting is required.

The performance gap of the no expediting policy can be large (up to 12.59%) when s_2 is small while the counterpart of the full expediting policy can be large (up to 11.61%) when s_1 is large or b_0 is small. Overall, the static policy in Table 4 is fairly effective ($\xi(\mathrm{S}) \leq 1.0\%$ for all cases). The static policy performs relatively better when (1) the unit backordering costs are large, (2) the unit expediting costs are small, (3) the unit ordering cost is small. The performance gap $\xi(\mathrm{S})$ is robust to the unit holding cost and the CoV of demands as inventory expediting is managed appropriately. Indeed, the good performance of the static policy compared with the other two heuristics reveals the value of inventory expediting and the necessity of managing inventory expediting appropriately in supply chains. Note that the performance gap $\xi(\mathcal{P})$ for $\mathcal{P} \in \{\mathrm{F}, \mathrm{N}, \mathrm{S}\}$ is not necessarily monotone in any cost parameter because the system cost of each policy may be increasing or decreasing in a cost parameter but such a property does not necessarily hold in $\xi(\mathcal{P})$ as $\xi(\mathcal{P}) = 100\% \times (C(\mathcal{P}) - C(\mathrm{Opt}))/C(\mathrm{Opt})$.

Through the numerical studies, we also identify when the full expediting policy and the no expediting policy are effective. The full expediting policy outperforms the no expediting policy and leads to an acceptable performance when b_1 is large, or the unit expediting costs s_1, s_2 are small. The no expediting policy is effective when $b_0, b_1, h, c, \sigma/\mu$ are small, or s_1, s_2 are large. It is intuitive that with small unit backordering costs and large unit expediting costs, we shall fulfill demands mainly with the on-hand inventory instead of the expedited inventory. Small values of h and c lead to a large ordering quantity and a high on-hand inventory level in each period. In this case, expediting may not be necessary and the no expediting policy is effective. A small σ/μ (CoV) means less demand uncertainty; then, inventory expediting may not be necessary as we can match demand and supply through regular ordering.

We conclude the findings of our numerical studies as follows. First, we numerically show the value of inventory expediting and the importance of managing expediting appropriately. Second, the static policy is effective under our settings. To further simplify the policy, we also identify when the full expediting and the no expediting policies are effective.

7. Concluding Remarks

We consider a periodic-review, single-stage inventory system with multiple demand classes, inventory expediting and a fixed replenishment leadtime. Compared with the existing literature, we are among the first to analyze such a complex system with multiple demand classes, inventory expediting, and a fixed replenishment leadtime. Due to its complexity, the optimal policy of such a system remains unknown in the existing literature. Hence, our results fill this gap.

To tackle this challenging problem, we first provide a new preservation property for the L$^\natural$-convexity under the minimization operator. This property might be useful in tackling other problems with multiple demand classes, e.g., dynamic matching. We then transform the state variables so that we can show the L$^\natural$-convexity of the value functions. We partially characterize the optimal policy for inventory ordering, expediting and allocation, and obtain various limited monotone properties based on the L$^\natural$-convexity. Specifically, we show that a state-dependent base stock policy is optimal for ordering decisions, a state-dependent threshold policy is optimal for inventory expediting, and a state-dependent rationing level policy is optimal for inventory allocation, respectively. In particular, we find that the optimal rationing levels are *independent* of backorder quantities of different classes. Moreover, we show that a *state-independent* rationing level policy is optimal when the leadtime is zero in the presence of replenishments during the planning horizon. It generalizes the result in Topkis (1968).

We also consider several special systems under which simple state-independent policies are optimal. For a system with two demand classes, we consider two cases and show the optimality of some simple policies based on the decomposition of the value functions for each case. For a system with one demand class, similar results are obtained. Inspired by the special cases, we propose three heuristic policies and numerically investigate when they can be effective. We also consider some other extensions in Appendix B, such as systems with convex backordering costs, Markov modulated demands, fixed ordering intervals, stochastic sequential leadtimes, and multiple stages.

In this paper, we make several assumptions in order to derive our results. For example, we assume a fixed replenishment leadtime, linear and convexly increasing expediting costs, etc. In future studies, one may consider a stochastic leadtime as in Kim et al. (2015) and analyze the optimal policy for that system. Another possible research avenue is to incorporate setup costs under our settings. However, the techniques developed in this paper are not applicable. It may be possible to develop heuristics for the ordering, expediting and allocation decisions in that case.

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References

Angelus, A., Ö. Özer. 2016. Knowledge You Can Act on: Optimal Policies for Assembly Systems with Expediting and Advance Demand Information. *Operations Research* 64(6) 1338-1371.

Arslan, H., S.C. Graves, T.A., Roemer. 2007. A single-product inventory model for multiple demand classes. *Management Science* 53(9) 1486-1500.

Chen, W., M. Dawande, G. Janakiraman. 2014. Fixed-dimensional stochastic dynamic programs: An approximation scheme and an inventory application. *Operations Research* 62(1) 81-103.

Chen, S., J. Xu, Y. Feng. 2010. A partial characterization of the optimal ordering/rationing policy for a periodic review system with two demand classes and backordering. *Naval Research Logistics* 57(4) 330-341.

Cohen, M. A., P. R. Kleindorfer, H. L. Lee. 1988. Service Constrained (s, S) Inventory Systems with Priority Demand Classes and Lost Sales. *Management Science* 34 (4) 482-499.

Dekker, R., M. J. Kleijn, P. J. de Rooij. 1998. A spare parts stocking policy based on equipment criticality. *International Journal of Production Economics* 56 69-77.

Dekker, R., R. M. Hill, M. J. Kleijn, R. H. Teunter. 2002. On the (S-1, S) lost sales inventory model with priority demand classes. *Naval Research Logistics* 49(6) 593-610.

Deshpande, V., M. A. Cohen. 2005. A nested threshold inventory rationing policy for multiple demand classes in inventory systems with replenishment. Working paper, Krannert School of Management, Purdue University, West Lafayette, IN.

Deshpande, V., M. A. Cohen, K. Donohue. 2003. A threshold inventory rationing policy for service-differentiated demand classes. *Management Science* 49(6) 683-703.

de Véricourt, F., F. Karaesmen, Y. Dallery. 2000. Dynamic scheduling in a make-to-stock system: A partial characterization of optimal policies. *Operations Research* 48(5) 811-819.

de Véricourt, F., F. Karaesmen, Y. Dallery. 2002. Optimal stock allocation for a capacitated supply system. *Management Science* 48(11) 1486-1501.

Evans, R. V. 1968. Sales and restocking policies in a single inventory system. *Management Science* 14(7) 463-473.

Fadıloğlu, M. M., Ö. Bulut. 2005. An embedded Markov chain approach to the analysis of inventory systems with backordering under rationing. Working paper, Department of Industrial Engineering, Bilkent University, Ankara, Turkey.

Frank, K. C., R. Q. Zhang, I. Duenyas. 2003. Optimal policies for inventory systems with priority demand classes. *Operations Research* 51(6) 993-1002.

Gayon, J., S. Benjaafar, F. de Véricourt. 2009. Using imperfect advance demand information in production-inventory systems with multiple customer classes. *Manufacturing and Service Operations Management* 11(1) 128-143.

Gong, X., X. Chao. 2013. Optimal control policy for capacitated inventory systems with remanufacturing. *Operations Research*, 61(3) 603-611.

Ha, A. Y. 1997a. Inventory rationing in a make-to-stock production system with several demand classes and lost sales. *Management Science* 43(8) 1093-1103.

Ha, A. Y. 1997b. Stock rationing policy for a make-to-stock production system with two priority classes and backordering. *Naval Research Logistics* 44(5) 457-472.

Huggins, E. L., T. L. Olsen. 2003. Supply chain management with guaranteed delivery. *Management Science* 49(9) 1154-1167.

Huh, W. T., G. Janakiraman. 2010. On the optimal policy structure in serial inventory systems with lost sales. *Operations Research* 58(2) 486-491.

Jain, A., Groenevelt, H., Rudi, N. 2011. Periodic review inventory management with contingent use of two freight modes with fixed costs. *Naval Research Logistics (NRL)* 58(4) 400-409.

Kaplan, A. 1969. Stock rationing. *Management Science* 15(5) 260-267.

Kim, C., D. Klabjan, D. Simchi-Levi. 2015. Optimal expediting policies for a serial inventory system with stochastic lead time. *Production and Operations Management* 24(10) 1524-1536.

Kouvelis, P., S. Y. Tang. 2012. On Optimal Expediting Policy for Supply Systems with Uncertain Lead-Times. *Production and Operations Management* 21(2) 309-330.

Lawson, D. G., E. L. Porteus. 2000. Multistage inventory management with expediting. *Operation Research* 48(6) 878-893.

Li, Q., P. Yu. 2014. Multimodularity and its applications in three stochastic dynamic inventory problems. *Manufacturing & Service Operations Management* 16 (3) 455-463.

Lu, Y., J. S. Song. 2005. Order-based cost optimization in assemble-to-order systems. *Operations Research* 53(1) 151-169.

Mamani, H., K. Moinzadeh. 2014. Lead time management through xxpediting in a continuous review inventory system. *Production and Operations Management* 23(1) 95-109.

Melchiors, P., R. Dekker, M. J. Kleijn. 2000. Inventory rationing in an (s, Q) inventory model with lost sales and two demand classes. *Journal of Operational Research Society* 51(1) 111-122.

Moinzadeh, K., S. Nahmias. 1988. A continuous review model for an inventory system with two supply modes. *Management Science* 34(6) 761-773.

Moinzadeh, K., C. P. Schmidt. 1991. An $(S-1, S)$ inventory system with emergency orders. *Operations Research* 39(2) 308-321.

Möllering, K.T., U. W. Thonemann. 2008. An optimal critical level policy for inventory systems with two demand classes. *Naval Research Logistics* 55(7) 632-642.

Muharremoğlu, A., J. N. Tsitsiklis. 2003. Dynamic leadtime management in supply chains. Technical report, Massachusetts Institute of Technology, Cambridge, MA.

Murota, K. 2000. Algorithms in discrete analysis. IEICE Transactions on Information System, E83-D, 344-352.

Murota, K. 2003. Discrete Convex Analysis. SIAM Monographs on Discrete Mathematics and Applications. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Murota, K. 2005. Note on multimodularity and L-convexity. *Mathematics of Operations Research* 30(3) 658-661.

Nahmias, S., S. Demmy. 1981. Operating characteristics of an inventory system with rationing. *Management Science* 27(11) 1236-1245.

Özsen, R., U.W. Thonemann. 2015. Determining optimal parameters for expediting policies. *Manufacturing & Service Operations Management* 17(1) 120-133.

Pang, Z., F. Y. Chen, Y. Feng. 2012. A note on the structure of joint inventory-pricing control with lead times. *Operations Research* 60(3) 581-587.

Tao, Y., et al. 2017. Inventory control policy for a periodic review system with expediting. *Applied Mathematical Modelling* 49 375-393.

Topkis, D. 1968. Optimal ordering and rationing policies in a nonstationary dynamic inventory model with n demand classes. *Management Science* 15(3) 160-176.

Veinott, A. F. 1965. Optimal policy in a dynamic, single product, nonstationary inventory model with several demand classes. *Operations Research* 13(5) 761-778.

Zhou, Y., X. Zhao. 2010. Optimal policies of an inventory system with multiple demand classes. *Tsinghua Science & Technology* 15(5) 498-508.

Zipkin, P. 2008. On the structure of lost-sales inventory model. *Operations Research* 56(4) 937-944.

Appendix A

First, we summarize the properties of L$^\natural$-convexity in the following lemma.

LEMMA 5. *L$^\natural$-convex functions have the following properties:*

(a) If $f(\boldsymbol{v})$ is L$^\natural$-convex, then $\psi(\boldsymbol{v},\zeta)=f(\boldsymbol{v}-\zeta\boldsymbol{e})$ is L$^\natural$-convex.

(b) If $g(\boldsymbol{v},\zeta)$ is L$^\natural$-convex, then $f(\boldsymbol{v})=\min_{\zeta\ge 0} g(\boldsymbol{v},\zeta)$ is L$^\natural$-convex.

(c) Let $\zeta(\boldsymbol{v})=\min\arg\min_{\zeta\ge 0} g(\boldsymbol{v},\zeta)$, then $\zeta(\boldsymbol{v})$ is nondecreasing in $\boldsymbol{v}$ and $\zeta(\boldsymbol{v}+\delta\boldsymbol{e})\le\zeta(\boldsymbol{v})+\delta$ for any $\delta>0$.

The proof is similar to that in Zipkin (2008) and is thus omitted. In particular, part (c) of Lemma 5 implies that the minimizer of an L$^\natural$-convex function is nondecreasing in $\boldsymbol{v}$, but the sensitivity is bounded by 1. ∎

In the following, we frequently use the notions $\phi\vee\zeta=\max\{\phi,\zeta\}$, $\phi\wedge\zeta=\min\{\phi,\zeta\}$, $\Phi\vee\Xi=(\phi_1\vee\zeta_1,\cdots,\phi_N\vee\zeta_N)$ and $\Phi\wedge\Xi=(\phi_1\wedge\zeta_1,\cdots,\phi_N\wedge\zeta_N)$, where $\Phi=(\phi_1,\cdots,\phi_N)$ and $\Xi=(\zeta_1,\cdots,\zeta_N)$.

Proof of Lemma 1

Let $\boldsymbol{x}=(x_1,\cdots,x_{n+m})$ and $\tilde{\boldsymbol{x}}=(\tilde{x}_1,\cdots,\tilde{x}_{n+m})$. Since $f(\boldsymbol{x})$ is L$^\natural$-convex, $\tilde{f}(\boldsymbol{x})=f(\boldsymbol{x})+\sum_{k=1}^{m+n}\gamma_k x_k$ must be L$^\natural$-convex as a linear function is L$^\natural$-convex and the summation preserves the L$^\natural$-convexity. Suppose that under the state $\boldsymbol{x}-\eta\boldsymbol{e}$ the optimal indices are (i^*,j^*) and the optimal solutions are $(y^*_{i^*},t^*_{n+j^*})$, whereas under the state $\tilde{\boldsymbol{x}}-\tilde{\eta}\boldsymbol{e}$ the optimal indices are (i^o,j^o) and the optimal solutions are $(y^o_{i^o},t_{n+j^o})$. For the ease of exposition, we define $\boldsymbol{y}^*_{i^*}=(y^*_{i^*},\cdots,y^*_{i^*})$, $\boldsymbol{y}^o_{i^o}=(y^o_{i^o},\cdots,y^o_{i^o})$, $\boldsymbol{x}_1=(x_1,\cdots,x_{n-1})$, and $\tilde{\boldsymbol{x}}_1=(\tilde{x}_1,\cdots,\tilde{x}_{n-1})$ as the $(n-1)$-dimensional vectors, and $\boldsymbol{t}^*_{n+j^*}=(t^*_{n+j^*},\cdots,t^*_{n+j^*})$, $\boldsymbol{t}^o_{n+j^o}=(t^o_{n+j^o},\cdots,t^o_{n+j^o})$, $\boldsymbol{x}_2=(x_{n+1},\cdots,x_{n+m-1})$, and $\tilde{\boldsymbol{x}}_2=(\tilde{x}_{n+1},\cdots,\tilde{x}_{n+m-1})$ as the $(m-1)$-dimensional vectors.

Since $y^*_{i^*},y^o_{i^o}\ge x_n$ and $t^*_{n+j^*},t^o_{n+j^o}\le x_{n+m}$, we have

$$\begin{cases} g(\boldsymbol{x}-\eta\boldsymbol{e})=\tilde{f}((\boldsymbol{x}_1\vee\boldsymbol{y}^*_{i^*},y^*_{i^*},\boldsymbol{x}_2\vee\boldsymbol{t}^*_{n+j^*},x_{n+m})-\eta\boldsymbol{e})+\zeta_t(t^*_{n+j^*}-\eta)+\zeta_y(y^*_{i^*}-\eta),\\ g(\tilde{\boldsymbol{x}}-\tilde{\eta}\boldsymbol{e})=\tilde{f}((\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^o_{i^o},y^o_{i^o},\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^o_{n+j^o},\tilde{x}_{n+m})-\eta\boldsymbol{e})+\zeta_t(t^o_{n+j^o}-\tilde{\eta})+\zeta_y(y^o_{i^o}-\tilde{\eta}). \end{cases}$$

Due to the L$^\natural$-convexity of $\tilde{f}$, based on the above two expressions, we obtain the following inequality:

$$\begin{aligned} & g(\boldsymbol{x}-\eta\boldsymbol{e})+g(\tilde{\boldsymbol{x}}-\tilde{\eta}\boldsymbol{e})\\ \ge&\tilde{f}(((\boldsymbol{x}_1\vee\boldsymbol{y}^*_{i^*})\vee(\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^o_{i^o}),y^*_{i^*}\vee y^o_{i^o},(\boldsymbol{x}_2\vee\boldsymbol{t}^*_{n+j^*})\vee(\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^o_{n+j^o}),x_{n+m}\vee\tilde{x}_{n+m})-\eta\boldsymbol{e})\\ &+\zeta_t(t^*_{n+j^*}\vee t^o_{n+j^o}-\eta\vee\tilde{\eta})+\zeta_y(y^*_{i^*}\vee y^o_{i^o}-\eta\vee\tilde{\eta})\\ &+\tilde{f}(((\boldsymbol{x}_1\vee\boldsymbol{y}^*_{i^*})\wedge(\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^o_{i^o}),y^*_{i^*}\wedge y^o_{i^o},(\boldsymbol{x}_2\vee\boldsymbol{t}^*_{n+j^*})\wedge(\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^o_{n+j^o}),x_{n+m}\wedge\tilde{x}_{n+m})-\eta\boldsymbol{e})\\ &+\zeta_t(t^*_{n+j^*}\wedge t^o_{n+j^o}-\eta\wedge\tilde{\eta})+\zeta_y(y^*_{i^*}\wedge y^o_{i^o}-\eta\wedge\tilde{\eta}). \end{aligned} \tag{24}$$

Since $(\boldsymbol{x}_1\vee\boldsymbol{y}^*_{i^*})\wedge(\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^o_{i^o})\ge(\boldsymbol{x}_1\wedge\boldsymbol{y}^*_{i^*})\wedge(\tilde{\boldsymbol{x}}_1\wedge\boldsymbol{y}^o_{i^o})$, $(\boldsymbol{x}_2\vee\boldsymbol{t}^*_{n+j^*})\wedge(\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^o_{n+j^o})\ge(\boldsymbol{x}_2\wedge\boldsymbol{t}^*_{n+j^*})\wedge(\tilde{\boldsymbol{x}}_2\wedge\boldsymbol{t}^o_{n+j^o})$, and $\tilde{f}(\boldsymbol{x})$ is nondecreasing in $x_1,\cdots,x_{n-1}$ and $x_{n+1},\cdots,x_{n+m-1}$, we have

$$\begin{aligned} &\tilde{f}(((\boldsymbol{x}_1\vee\boldsymbol{y}^*_{i^*})\wedge(\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^o_{i^o}),y^*_{i^*}\wedge y^o_{i^o},(\boldsymbol{x}_2\vee\boldsymbol{t}^*_{n+j^*})\wedge(\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^o_{n+j^o}),x_{n+m}\wedge\tilde{x}_{n+m})-\eta\boldsymbol{e})\\ \ge&\tilde{f}(((\boldsymbol{x}_1\wedge\boldsymbol{y}^*_{i^*})\wedge(\tilde{\boldsymbol{x}}_1\wedge\boldsymbol{y}^o_{i^o}),y^*_{i^*}\wedge y^o_{i^o},(\boldsymbol{x}_2\wedge\boldsymbol{t}^*_{n+j^*})\wedge(\tilde{\boldsymbol{x}}_2\wedge\boldsymbol{t}^o_{n+j^o}),x_{n+m}\wedge\tilde{x}_{n+m})-\eta\boldsymbol{e}). \end{aligned}$$

Then, based on the inequality (24),

$$\begin{aligned}g(\boldsymbol{x}-\eta\boldsymbol{e})+g(\tilde{\boldsymbol{x}}-\tilde{\eta}\boldsymbol{e})\geq&\tilde{f}((\boldsymbol{x}_1\vee\tilde{\boldsymbol{x}}_1\vee\boldsymbol{y}^*_{i^*}\vee\boldsymbol{y}^o_{i^o},y^*_{i^*}\vee y^o_{i^o},\boldsymbol{x}_2\vee\tilde{\boldsymbol{x}}_2\vee\boldsymbol{t}^*_{n+j^*}\vee\boldsymbol{t}^o_{n+j^o},x_{n+m}\vee\tilde{x}_{n+m})-\eta\boldsymbol{e})\\&+\zeta_t(t^*_{n+j^*}\vee t^o_{n+j^o}-\eta\vee\tilde{\eta})+\zeta_y(y^*_{i^*}\vee y^o_{i^o}-\eta\vee\tilde{\eta})\\&+\tilde{f}((\boldsymbol{x}_1\wedge\tilde{\boldsymbol{x}}_1\wedge\boldsymbol{y}^*_{i^*}\wedge\boldsymbol{y}^o_{i^o},y^*_{i^*}\wedge y^o_{i^o},\boldsymbol{x}_2\wedge\tilde{\boldsymbol{x}}_2\wedge\boldsymbol{t}^*_{n+j^*}\wedge\boldsymbol{t}^o_{n+j^o},x_{n+m}\wedge\tilde{x}_{n+m})-\eta\boldsymbol{e})\\&+\zeta_t(t^*_{n+j^*}\wedge t^o_{n+j^o}-\eta\wedge\tilde{\eta})+\zeta_y(y^*_{i^*}\wedge y^o_{i^o}-\eta\wedge\tilde{\eta}).\end{aligned}\tag{25}$$

Since $x_n\vee\tilde{x}_n\leq y^*_{i^*}\vee y^o_{i^o}\leq\min\{x_1\vee\tilde{x}_1,t^*_{n+j^*}\vee t^o_{n+j^o}\}$ and $x_{n+1}\vee\tilde{x}_{n+1}\leq t^*_{n+j^*}\vee t^o_{n+j^o}\leq x_{n+m}\vee\tilde{x}_{n+m}$, then $y^*_{i^*}\vee y^o_{i^o}$ and $t^*_{n+j^*}\vee t^o_{n+j^o}$ are feasible solutions to $g(\boldsymbol{x}\vee\tilde{\boldsymbol{x}}-\eta\vee\tilde{\eta}\boldsymbol{e})$. Similarly, $y^*_{i^*}\wedge y^o_{i^o}$ and $t^*_{n+j^*}\wedge t^o_{n+j^o}$ are feasible solutions to $g(\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}-\eta\wedge\tilde{\eta}\boldsymbol{e})$. As a result, based on the inequality (25), we have

$$g(\boldsymbol{x}-\eta\boldsymbol{e})+g(\tilde{\boldsymbol{x}}-\tilde{\eta}\boldsymbol{e})\geq g(\boldsymbol{x}\vee\tilde{\boldsymbol{x}}-\eta\vee\tilde{\eta}\boldsymbol{e})+g(\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}-\eta\wedge\tilde{\eta}\boldsymbol{e})$$

because the feasible solutions lead to a higher value of the function $g(\boldsymbol{x}\vee\tilde{\boldsymbol{x}}-\eta\vee\tilde{\eta}\boldsymbol{e})+g(\boldsymbol{x}\wedge\tilde{\boldsymbol{x}}-\eta\wedge\tilde{\eta}\boldsymbol{e})$. ■

Proof of Lemma 2

In period t, we let $(\boldsymbol{w},\boldsymbol{x})$ be the initial state and q^* be the optimal ordering quantity. After observing the realized demand $\boldsymbol{D}$, we denote by $\boldsymbol{a}^*=(a^*_0,\cdots,a^*_n)$ the optimal quantities of inventory allocated to demand classes $0,\cdots,n$, and $\boldsymbol{o}^*_t=(o^*_{1,t},\cdots,o^*_{l,t})$ the optimal expediting quantities from leadtime positions $1,\cdots,l$. Then,

$$\begin{aligned}f_t(\boldsymbol{w},\boldsymbol{x})=&cq^*+g_t(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\\=&cq^*+\sum_{k=0}^n b_k(w_k+D_k-a^*_k)+h(x_0+x_1-\sum_{k=0}^n a^*_k+\sum_{i=2}^l o^*_i)+\sum_{i=1}^l s_io^*_i+\beta f_{t+1}(\boldsymbol{w}+\boldsymbol{D}-\boldsymbol{a}^*,\hat{\boldsymbol{x}}),\end{aligned}$$

where $\hat{\boldsymbol{x}}=(\hat{x}_0,\cdots,\hat{x}_{l-1})$, $\hat{x}_0=x_0+x_1-\sum_{k=0}^n a^*_k+\sum_{i=2}^l o^*_i$, and $\hat{x}_i=x_{i+1}-o^*_{i+1}$ for $i=1,\cdots,l-1$.

To prove Lemma 2 (1), we consider another initial state $(\boldsymbol{w}+\delta_1(\boldsymbol{e}_{m+1}-\boldsymbol{e}_{j+1}),\boldsymbol{x})$ in period t, where $0\leq\delta_1\leq w_j$ and $n\geq m>j\geq 0$. The state results from the decision in period $t-1$ such that serving δ_1 more backorders of class j and accordingly δ_1 fewer backorders of class m. Hence, the costs in periods $1,\cdots,t-2$ are kept the same but the cost in period $t-1$ increases by $(b_m-b_j)\delta_1$. Hence, we shall prove that

$$f_t(\boldsymbol{w}+\delta_1(\boldsymbol{e}_{m+1}-\boldsymbol{e}_{j+1}),\boldsymbol{x})+\frac{\delta_1}{\beta}(b_m-b_j)\leq f_t(\boldsymbol{w},\boldsymbol{x})\tag{26}$$

for $m>j$ by induction. The property holds in period $T+l+1$ since $b_m<b_j$. Suppose that it holds for period $t+1$, it suffices to show that the property holds in period t as well.

Let $\bar{\delta}_1=\min\{a^*_j,\delta_1\}$. Then, $\bar{\delta}_1\leq\delta_1$. Under the state $(\boldsymbol{w}+\delta_1(\boldsymbol{e}_{m+1}-\boldsymbol{e}_{j+1}),\boldsymbol{x})$, q^* is a feasible ordering decision and $\bar{\boldsymbol{a}}=\boldsymbol{a}^*-\bar{\delta}_1(\boldsymbol{e}_{j+1}-\boldsymbol{e}_{m+1})$ is a feasible inventory allocation policy. Hence,

$$\begin{aligned}&f_t(\boldsymbol{w},\boldsymbol{x})-\left[f_t(\boldsymbol{w}+\delta_1(\boldsymbol{e}_{m+1}-\boldsymbol{e}_{j+1}),\boldsymbol{x})+\frac{\delta_1}{\beta}(b_m-b_j)\right]\\\geq&-(b_m-b_j)\frac{\delta_1}{\beta}-(b_m-b_j)(\delta_1-\bar{\delta}_1)+\beta f_{t+1}(\boldsymbol{w}+\boldsymbol{D}-\boldsymbol{a}^*,\hat{\boldsymbol{x}})-\beta f_{t+1}(\boldsymbol{w}+(\delta_1-\bar{\delta}_1)(\boldsymbol{e}_{m+1}-\boldsymbol{e}_{j+1})+\boldsymbol{D}-\boldsymbol{a}^*,\hat{\boldsymbol{x}})\\\geq&-(b_m-b_j)\frac{\delta_1}{\beta}-(b_m-b_j)(\delta_1-\bar{\delta}_1)+(\delta_1-\bar{\delta}_1)(b_m-b_j)\geq 0\end{aligned}$$

due to $b_m<b_j$ for $m>j$ and the inductive assumption.

Since substituting any quantity of the backorders from class m by the same quantity of backorders from class j in period $t-1$ would incur less total cost, it follows that it is always optimal to satisfy higher-priority demands first. Specifically, we should first fulfill class 0 demand and then class 1 demand and so on. We thus have completed the proofs of Lemma 2 (1).

To prove Lemma 2 (2), we consider another initial state $(\boldsymbol{w}, \boldsymbol{x}+\delta_2(\boldsymbol{e}_{r+1}-\boldsymbol{e}_{i+1}))$ in period t, where $0 \le \delta_2 \le x_i$ and $r > i$. The state results from the decision in period $t-1$ such that expediting δ_2 more units of inventory from leadtime position $i+1$ and δ_2 less from leadtime position $r+1$. Hence, we shall prove that

$$f_t(\boldsymbol{w}, \boldsymbol{x}+\delta_2(\boldsymbol{e}_{r+1}-\boldsymbol{e}_{i+1})) + \frac{\delta_2}{\beta}(s_{i+1}-s_{r+1}) \le f_t(\boldsymbol{w}, \boldsymbol{x}) \tag{27}$$

for $l-1 \ge r > i \ge 0$ by induction. Let $\bar{\delta}_2 = \min\{o_i^*, \delta_2\}$. Then, $\bar{\delta}_2 \le \delta_2$. Under the state $(\boldsymbol{w}, \boldsymbol{x}+\delta_2(\boldsymbol{e}_{r+1}-\boldsymbol{e}_{i+1}))$, q^* is a feasible ordering decision and $\bar{\boldsymbol{o}} = \boldsymbol{o}^* - \bar{\delta}_2(\boldsymbol{e}_{i+1}-\boldsymbol{e}_{r+1})$ is a feasible inventory expediting policy. Suppose that the property holds in period $t+1$. Then,

$$\begin{aligned}
&f_t(\boldsymbol{w}, \boldsymbol{x}) - [f_t(\boldsymbol{w}, \boldsymbol{x}+\delta_2(\boldsymbol{e}_{r+1}-\boldsymbol{e}_{i+1})) + \frac{\delta_2}{\beta}(s_{i+1}-s_{r+1})]\\
\ge& \frac{\delta_2}{\beta}(s_{r+1}-s_{i+1}) - \bar{\delta}_2(s_r - s_i) + \beta f_{t+1}(\boldsymbol{w}+\boldsymbol{D}-\boldsymbol{a}^*, \hat{\boldsymbol{x}}) - \beta f_{t+1}(\boldsymbol{w}+\boldsymbol{D}-\boldsymbol{a}^*, \hat{\boldsymbol{x}} + (\delta_2-\bar{\delta}_2)(\boldsymbol{e}_r-\boldsymbol{e}_i))\\
\ge& \frac{\delta_2}{\beta}(s_{r+1}-s_{i+1}) - \bar{\delta}_2(s_r-s_i) - (\delta_2-\bar{\delta}_2)(s_r-s_i) \ge 0
\end{aligned}$$

duo to $s_{r+1}-s_{i+1} \ge s_r - s_i \ge 0$, $\beta \le 1$, and the inductive assumption. In addition, since the positive on-hand inventory would incur holding costs, it is always better to serve demands with on-hand inventory first than to serve them with expedited inventory from stage 1.

The results imply that substituting any quantity of inventory at leadtime position r by the same quantity of inventory at leadtime position i for any $i < r$ would incur less total cost. Hence, it is always optimal to serve demands with on-hand inventory first and, if expediting is necessary, it is optimal to expedite inventory from leadtime position i if and only if there is no inventory at leadtime positions $0, \cdots, i-1$. We thus have completed the proof of Lemma 2 (2). ■

Proof of Lemma 3

Suppose that we expedite up to leadtime position i and set its net inventory level be t_i after inventory expediting and allocation, and we also allocate up to demand class j and set the negatively aggregated backorder quantity of classes $j, \cdots, n$ be y_j. Based on the definition of the transformed state variables,

$$\begin{cases}
x_l = v_l - v_{l-1}, & \\
w_k + D_k - a_k = 0, & k = 0, \cdots, i-1,\\
w_j + D_j - a_j = \bar{z}_{j+1} - y_j, & \\
w_k + D_k - a_k = \bar{z}_{k+1} - \bar{z}_k, & k = j+1, \cdots, n,\\
x_0 + \sum_{r=1}^{l} o_r - \sum_{k=0}^{n} a_k = t_i - y_j, & \\
o_r = \bar{v}_r - \bar{v}_{r-1}, & r = 1, \cdots, i-1,\\
o_i = t_i - \bar{v}_{i-1}, & \\
o_r = 0, & r = i+1, \cdots, l.
\end{cases}$$

It follows that

$$\sum_{k=0}^{n} b_k(w_k+D_k-a_k)+h(x_0+\sum_{r=1}^{l} o_r-\sum_{k=0}^{n} a_k)+\sum_{r=1}^{l} s_i o_i=\sum_{k=j+1}^{n}\hat{b}_k\bar{z}_k-b_jy_j+h(t_i-y_j)+\sum_{r=0}^{i-1}\hat{s}_r\bar{v}_r+s_it_i.$$

For the initial state of the next period $t+1$, we discuss it in detail as follows. Based on Lemma 2, if we decide to allocate up to demand class j then we must have fulfilled all demands of classes $0,\cdots,j-1$. In this case, we shall have $z_{0+}=\cdots=z_{j+}=y_j$; see (5). Moreover, the allocation to demand classes $0,\cdots,j$ won't change the variables $z_{(j+1)+},\cdots,z_{n+}$; see (5). Similarly, if we decide to expedite up to leadtime position i, then there is no inventory at lower leadtime positions. In this case, we shall have $v_{0+}=\cdots=v_{(i-2)+}=t_i$; see (6). Moreover, the expediting decision for leadtime positions $1,\cdots,i$ won't change the variables $v_{(i-1)+},\cdots,v_{(l-1)+}$; see (6). Hence, if we decide to expedite up to leadtime position i and allocate up to demand class j, the optimal expediting and allocation policies are given by $F_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ in (9). When it is infeasible to allocate inventory to demand class j, we have $F_{ij}\equiv\infty$ and we would not choose such a pair of (i,j). Therefore,

$$\bar{g}_t(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})=\min_{j=0,\cdots,n,i=1,\cdots,l} F_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$$

is the optimal cost given $(\boldsymbol{z},\boldsymbol{v},v_l)$ and the realized demand $\boldsymbol{D}$. Then, it is easy to observe that the optimization problem in (8) and (9) is equivalent to the optimization problem in (2).

Since

$$\bar{f}_t(\boldsymbol{z},\boldsymbol{v})=\min_{v_l\geq v_{l-1}}\mathbb{E}[c(v_l-v_{l-1})+\bar{g}_t(z_n-D_n,\cdots,z_0-\sum_{k=0}^{n}D_k,(\boldsymbol{v},v_l)-\sum_{k=0}^{n}D_k\boldsymbol{e})],$$

it follows that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$ is the optimal cost from period t and onward, which is equivalent to the optimal cost function $f_t(\boldsymbol{w},\boldsymbol{x})$ in (1).

We then show the monotone properties. To show that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Phi'}{\beta}\boldsymbol{z}$ is nondecreasing in z_j for $j=1,\cdots,n$, we shall show that $\bar{f}_t(\boldsymbol{z}-\delta\boldsymbol{e}_{n+1-j},\boldsymbol{v})+\frac{\Phi'}{\beta}(\boldsymbol{z}-\delta\boldsymbol{e}_{n+1-j})\leq\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Phi'}{\beta}\boldsymbol{z}$, i.e., $\bar{f}_t(\boldsymbol{z}-\delta\boldsymbol{e}_{n+1-j},\boldsymbol{v})-\frac{\delta}{\beta}(b_{j-1}-b_j)\leq\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$. According to the definition of z_j, decreasing z_j is equivalent to increasing w_j and decreasing w_{j-1}. Then, based on the inequality (26), $\bar{f}_t(\boldsymbol{z}-\delta\boldsymbol{e}_{n+1-j},\boldsymbol{v})-\frac{\delta}{\beta}(b_{j-1}-b_j)\leq\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$ must hold. Hence, $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Phi'}{\beta}\boldsymbol{z}$ is nondecreasing in z_j for $j=1,\cdots,n$. Similarly, to show that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Theta'}{\beta}\boldsymbol{v}$ is nondecreasing in v_i for $i=0,\cdots,l-2$, we shall show that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v}-\delta\boldsymbol{e}_{i+1})+\frac{\Theta'}{\beta}(\boldsymbol{v}-\delta\boldsymbol{e}_{i+1})\leq\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Theta'}{\beta}\boldsymbol{v}$, i.e., $\bar{f}_t(\boldsymbol{z},\boldsymbol{v}-\delta\boldsymbol{e}_{i+1})-\frac{\delta}{\beta}(s_{i+2}-s_{i+1})\leq\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$. As decreasing v_i is equivalent to decreasing x_i and increasing x_{i+1}. Based on the inequality (27), we conclude that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})+\frac{\Theta'}{\beta}\boldsymbol{v}$ is nondecreasing in v_i for $i=0,\cdots,l-2$. ■

Proof of Theorem 1

Part (1). We show the result by induction. Note that $\bar{f}_{T+l+1},\bar{g}_{T+l+1}$ are L$^\natural$-convex. Suppose that $\bar{f}_{t+1}$ is L$^\natural$-convex. Then, it suffices to show that $\bar{f}_t$ and $\bar{g}_t$ are L$^\natural$-convex.

Note that

$$\bar{f}_t(\boldsymbol{z}-\eta\boldsymbol{e},\boldsymbol{v}-\eta\boldsymbol{e})=\min_{v_l\geq v_{l-1}}\mathbb{E}\left[c(v_l-v_{l-1})+\bar{g}_t(z_n-\eta-D_n,\cdots,z_0-\eta-\sum_{k=0}^{n}D_k,(\boldsymbol{v},v_l)-\eta\boldsymbol{e}-\sum_{k=0}^{n}D_k\boldsymbol{e})\right],$$

and $\bar{g}_t(\bar{\boldsymbol{z}}-\eta\boldsymbol{e},\bar{\boldsymbol{v}}-\eta\boldsymbol{e})=\min_{j=0,\cdots,n,\,i=1,\cdots,l}F_{ij,t}(\bar{\boldsymbol{z}}-\eta\boldsymbol{e},\bar{\boldsymbol{v}}-\eta\boldsymbol{e})$, where for $j=0,\cdots,n$, $i=1,\cdots,l$

$$F_{ij,t}(\bar{\boldsymbol{z}}-\eta\boldsymbol{e},\bar{\boldsymbol{v}}-\eta\boldsymbol{e})=\min_{\bar{z}_j\leq y_j\leq\min\{\bar{z}_{j+1},t_i\},\bar{v}_{i-1}\leq t_i\leq\bar{v}_i}\widetilde{F}_{ij,t}$$

and

$$\begin{aligned}\widetilde{F}_{ij,t} \equiv& \beta \bar{f}_{t+1}(\bar{z}_n - \eta, \cdots, \bar{z}_{j+1} - \eta, \overbrace{y_j - \eta, \cdots, y_j - \eta}^{j+1}, \overbrace{t_i - \eta, \cdots, t_i - \eta}^{i-1}, \bar{v}_i - \eta, \cdots, \bar{v}_l - \eta) \\ &+ \sum_{k=0}^{i-1} \hat{s}_k(\bar{v}_k - \eta) + s_i(t_i - \eta) + \sum_{k=j+1}^{n} \hat{b}_k(\bar{z}_k - \eta) - b_j(y_j - \eta) + h(t_i - y_j).\end{aligned}$$

We shall show that $\widetilde{F}_{ij,t}$ has the same properties with the term $\tilde{f} + \bar{\gamma}_t t_{n+j} + \bar{\gamma}_y y_i$ in Lemma 1, and based on which we show the L$^\natural$-convexity of $\bar{g}_t$ based on Lemma 1. We rewritten $\widetilde{F}_{ij,t}$ as follows (noting that $\bar{z}_n \geq \cdots \geq \bar{z}_0 \leq \bar{v}_0 \leq \bar{v}_1 \leq \cdots \bar{v}_l$, $\bar{z}_j \leq y_j \leq \min\{\bar{z}_{j+1}, t_i\}$, $\bar{v}_{i-1} \leq t_i \leq \bar{v}_i$):

$$\widetilde{F}_{ij,t} = \tilde{f}_t(\bar{z}_n \vee y_j - \eta, \cdots, \bar{z}_0 \vee y_j - \eta, t_i \vee \bar{v}_1 - \eta, \cdots, t_i \vee \bar{v}_l - \eta) + ht_i - hy_j + \sum_{k=0}^{l-1} \hat{s}_k(\bar{v}_k - \eta)$$

where

$$\begin{aligned}&\tilde{f}_t(\bar{z}_n \vee y_j - \eta, \cdots, \bar{z}_0 \vee y_j - \eta, t_i \vee \bar{v}_1 - \eta, \cdots, t_i \vee \bar{v}_l - \eta) \\ =& \beta \bar{f}_{t+1}(\bar{z}_n \vee y_j - \eta, \cdots, \bar{z}_0 \vee y_j - \eta, t_i \vee \bar{v}_1 - \eta, \cdots, t_i \vee \bar{v}_l - \eta) + \sum_{k=0}^{n} \hat{b}_k(\bar{z}_k \vee y_j - \eta) - \sum_{k=0}^{l-1} \hat{s}_k(t_i \vee \bar{v}_k - \eta).\end{aligned}$$

Based on Lemma 3, $\beta \bar{f}_{t+1}(\boldsymbol{z}, \boldsymbol{u}) + \hat{b}_j z_j$ is nondecreasing in z_j for $j = 1, \cdots, n$ and $\beta \bar{f}_{t+1}(\boldsymbol{z}, \boldsymbol{u}) - \hat{s}_{i+1} v_i$ is nondecreasing in v_i for $i = 0, \cdots, l-2$. Hence, we conclude that $\tilde{f}_t$ is nondecreasing in $(\bar{z}_n \vee y_j, \cdots, \bar{z}_0 \vee y_j, \bar{v}_1 \vee t_i, \cdots, \bar{v}_l \vee t_i)$, which is similar to $\tilde{f}$ in Lemma 1. Moreover, $ht_i - hy_j$ corresponds to $\zeta_t t_{n+j} + \zeta_y y_i$ in Lemma 1. We also have $\bar{v}_l \geq \cdots \geq \bar{v}_0 \geq \bar{z}_0 \leq \bar{z}_1 \leq \cdots \leq \bar{z}_n$ by our definition. These lead to the result that $\bar{g}_t$ is L$^\natural$-convex in $(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ by Lemma 1. Furthermore, notice that $v_l \geq v_{l-1}$ is a sublattice and, from Theorem 2.7.6 of Topkis (1998), we know that submodularity can be preserved under the minimization over a sublattice. As a result, $\bar{f}_t$ is L$^\natural$-convex in $(\boldsymbol{z}, \boldsymbol{v})$. We thus have completed the inductive proof.

Part (2). Since

$$\begin{aligned}(Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}), R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})) = \min \operatorname*{arg\,min}_{\bar{z}_j \leq y_j \leq \min\{\bar{z}_{j+1}, t_i\}, \bar{v}_{i-1} \leq t_i \leq \bar{v}_i} &[\sum_{k=0}^{i-1} \hat{s}_k \bar{v}_k + s_i t_i + \sum_{k=j+1}^{n} \hat{b}_k \bar{z}_k - b_j y_j + h(t_i - y_j) \\ &+ \beta \bar{f}_{t+1}(\bar{z}_n, \cdots, \bar{z}_{j+1}, \overbrace{y_j, \cdots, y_j}^{j+1}, \overbrace{t_i, \cdots, t_i}^{i-1}, \bar{v}_i, \cdots, \bar{v}_{l-1}, \bar{v}_l)],\end{aligned}$$

$R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are only related to $\bar{z}_n, \cdots, \bar{z}_{j+1}, y_j$ and $t_i, \bar{v}_i, \cdots, \bar{v}_l$. Note that y_j is within the range $[\bar{z}_j, \bar{z}_{j+1}]$ and t_i is within the range $[\bar{v}_{i-1}, \bar{v}_i]$. Hence, $R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are independent of $\bar{z}_{j-1}, \cdots, \bar{z}_0$ and $\bar{v}_0, \cdots, \bar{v}_{i-2}$. In addition, according to Lemma 5 (c), the L$^\natural$-convexity of $\bar{f}_{t+1}$ implies that $R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are nondecreasing in $(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ (more specifically, $\bar{z}_n, \cdots, \bar{z}_j$ and $\bar{v}_{i-1}, \cdots, \bar{v}_l$ since $R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are independent of the other items).

Part (3). Since $R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are nondecreasing in $\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}$, it follows that $i_t^*(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $j_t^*(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ are also nondecreasing in $(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$.

Parts (4) and (5). Due to the sequential property of inventory expediting and allocation, the optimal solutions $R_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ and $Y_{ij,t}(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$ directly imply the optimal policies described in Theorem 1.

Part (6). From Part (1), we know that $\bar{g}_t$ is L$^\natural$-convex. Then based on Lemma 5 (c), we know that $S_t^*(\boldsymbol{z}, \boldsymbol{v})$ is nondecreasing in $(\boldsymbol{z}, \boldsymbol{v})$ and satisfies $S_t^*(\boldsymbol{z} + \delta \boldsymbol{e}, \boldsymbol{v} + \delta \boldsymbol{e}) \leq S_t^*(\boldsymbol{z}, \boldsymbol{v}) + \delta$. Then clearly the inequalities $S_t^*(\boldsymbol{z}, \boldsymbol{v}) \leq$

$S_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \le S_t^*(\boldsymbol{z},\boldsymbol{v})+\delta$ must be valid. With $S_t^*(\boldsymbol{z},\boldsymbol{v}) = q_t^*(\boldsymbol{z},\boldsymbol{v})+v_{l-1}$, we have that $q_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_l) = S_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_l)-\delta-v_{l-1} \ge S_t^*(\boldsymbol{z},\boldsymbol{v})-\delta-v_{l-1} = q_t^*(\boldsymbol{z},\boldsymbol{v})-\delta$. Clearly, $q_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_l) \ge q_t^*(\boldsymbol{z},\boldsymbol{v})-\delta$ holds. Due to the inequality $S_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \le S_t^*(\boldsymbol{z},\boldsymbol{v})+\delta$, we obtain $q_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) = S_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e})-\delta-v_{l-1} \le S_t^*(\boldsymbol{z},\boldsymbol{v})-v_{l-1} = q_t^*(\boldsymbol{z},\boldsymbol{v})$, i.e., $q_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \le q_t^*(\boldsymbol{z},\boldsymbol{v})$. ■

Proof of Theorem 2

By Theorem 1, we have $S_t^*(\boldsymbol{w},\boldsymbol{x}+\delta(-\boldsymbol{e}_{i+1}+\boldsymbol{e}_i)) = S_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i) \ge S_t^*(\boldsymbol{z},\boldsymbol{v}) = S_t^*(\boldsymbol{w},\boldsymbol{x})$ for $i=1,\cdots,l-1$ and $q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_l) = q_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_l) \ge q_t^*(\boldsymbol{w},\boldsymbol{x})-\delta$. For $i=1,\cdots,l-1$, we can obtain $q_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i) = S_t^*(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i)-v_{l-1} = q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta(\boldsymbol{e}_i-\boldsymbol{e}_{i+1})) \ge q_t^*(\boldsymbol{w},\boldsymbol{x})$, which means that decreasing any quantity of inventory at leadtime position i and increasing the same quantity of inventory at leadtime position $i-1$ would lead to a larger value of the optimal ordering quantity. It follows that $q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1) \ge \cdots \ge q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_l)$. Hence, we obtain that $q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1)-q_t^*(\boldsymbol{w},\boldsymbol{x}) \ge \cdots \ge q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_l)-q_t^*(\boldsymbol{w},\boldsymbol{x}) \ge -\delta$.

Because that it is always to fulfill class 0 demand with, if possible, the on-hand inventory first, decreasing any quantity of class 0 demand is equivalent to increasing the same quantity of on-hand inventory, i.e., $q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1) = q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_1,\boldsymbol{x})$. Also, $q_t^*(\boldsymbol{w}+\delta(\boldsymbol{e}_j-\boldsymbol{e}_{j-1}),\boldsymbol{x}) = q_t^*(\boldsymbol{z}-\delta\boldsymbol{e}_j,\boldsymbol{v}) \le q_t^*(\boldsymbol{z},\boldsymbol{v}) = q_t^*(\boldsymbol{w},\boldsymbol{x})$ for $j=2,\cdots,n+1$, where the inequality holds due to the facts that $q_t^*(\boldsymbol{z},\boldsymbol{v}) = S_t^*(\boldsymbol{z},\boldsymbol{v})-v_{l-1}$ and $S_t^*(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $\boldsymbol{z}$ and $\boldsymbol{v}$. It indicates that substituting any quantity of class $j-1$ backorders with the same quantity of class j backorders would lead to a smaller value of optimal ordering quantity, i.e., $q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_1,\boldsymbol{x}) \le q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_2,\boldsymbol{x}) \le \cdots \le q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_{n+1},\boldsymbol{x})$. In addition, by Theorem 1, we have $q_t^*(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \le q_t^*(\boldsymbol{z},\boldsymbol{v})$. Then $q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_{n+1},\boldsymbol{x}) = q_t^*(\boldsymbol{z}+\delta\boldsymbol{e},v+\delta\boldsymbol{e}) \le q_t^*(\boldsymbol{z},\boldsymbol{v}) = q_t^*(\boldsymbol{w},\boldsymbol{x})$. As a result, we obtain $-\delta \le q_t^*(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1)-q_t^*(\boldsymbol{w},\boldsymbol{x}) = q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_1,\boldsymbol{x})-q_t^*(\boldsymbol{w},\boldsymbol{x}) \le q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_2,\boldsymbol{x})-q_t^*(\boldsymbol{w},\boldsymbol{x}) \le \cdots \le q_t^*(\boldsymbol{w}-\delta\boldsymbol{e}_{n+1},\boldsymbol{x})-q_t^*(\boldsymbol{w},\boldsymbol{x}) \le 0$.

Due to L$^\natural$-convexity of $\bar{g}_t$, it follows that $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}), R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ are nondecreasing in $(\bar{z}_n,\cdots,\bar{z}_j,\bar{v}_{i-1},\cdots,\bar{v}_l)$ and are independent of $(\bar{z}_{j-1},\cdots,\bar{z}_0,\bar{v}_0,\cdots,\bar{v}_{i-2})$. Note that we only fulfill class j demand if we have fulfilled all demands of classes $0,\cdots,j-1$, which implies that $a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_k,\bar{\boldsymbol{x}}) = a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}-\delta\boldsymbol{e}_1)$ for $k=1,\cdots,j$. Hence,

$$a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})-a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}) = \cdots = a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_j,\bar{\boldsymbol{x}})-a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}) \le 0.$$

By Theorem 1, $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ is nondecreasing in $\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}$ (note that $a_{j,t}^*(\bar{\boldsymbol{z}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{v}}) = Y_{ij,t}(\bar{\boldsymbol{z}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{v}})-\bar{z}_j-\delta \ge Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-\bar{z}_j-\delta = a_{j,t}^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-\delta$). Then $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}) \le Y_{ij,t}(\bar{z}_n+\delta,\cdots,\bar{z}_j+\delta,\bar{\boldsymbol{v}}+\delta\boldsymbol{e}) \le Y_{ij,t}(\bar{z}_n,\cdots,\bar{z}_j,\bar{\boldsymbol{v}})+\delta$, which implies that $a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}) \le a_{j,t}^*(\bar{\boldsymbol{w}}-\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}})+\delta \le a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})+\delta$. Then it is easy to obtain that $a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}}) \le a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})+\delta$. In addition, $Y_{ij,t}(\bar{\boldsymbol{z}}+\delta\boldsymbol{e}_k,\bar{\boldsymbol{v}}) \ge Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ for $k>j+1$, which implies that $a_{j,t}^*(\bar{\boldsymbol{w}}-\delta(\boldsymbol{e}_k-\boldsymbol{e}_{k-1}),\bar{\boldsymbol{x}}) \ge a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})$. This inequality is equivalent to $a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{k-1},\bar{\boldsymbol{x}}) \ge a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_k,\bar{\boldsymbol{x}})$. Moreover, $a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{n+1},\bar{\boldsymbol{x}}) = Y_{ij,t}(\bar{\boldsymbol{z}}-\delta\boldsymbol{e},\bar{\boldsymbol{v}}-\delta\boldsymbol{e})-(\bar{z}_j-\delta) \le Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-(\bar{z}_j-\delta) = a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})+\delta$. As a result, we have

$$\delta \ge a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}})-a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}) \ge \cdots \ge a_{j,t}^*(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{n+1},\bar{\boldsymbol{x}})-a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}) \ge 0.$$

Similarly, $Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}+\delta\boldsymbol{e}_k) \ge Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ implies that $a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta(\boldsymbol{e}_k-\boldsymbol{e}_{k+1})) \ge a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})$ for $k=1,\cdots,l$. In addition, $a_{j,t}^*(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_{l+1}) = a_{j,t}^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}+\delta\boldsymbol{e}_{l+1}) = Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}+\delta\boldsymbol{e}_{l+1})-\bar{z}_j \ge Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-\bar{z}_j \ge a_{j,t}^*(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}}) =$

$a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})$. Note that $a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)=Y_{ij,t}(\bar{z}_0-\delta,\cdots,\bar{z}_j-\delta,\bar{z}_{j+1},\cdots,\bar{z}_n,\bar{\boldsymbol{v}})-\bar{z}_j+\delta\le Y_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-\bar{z}_j+\delta=a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})+\delta$, i.e., $a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)\le a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})+\delta$. Also $a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)\le a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)$ because of the proved result that $a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0$. Hence, we obtain

$$\delta\ge a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge\cdots\ge a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_{l+1})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0.$$

Note that $a^*_{j,t}(\bar{\boldsymbol{w}}-\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})=a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)$. Then $a^*_{j,t}(\bar{\boldsymbol{w}}-\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\le\delta$, which is equivalent to $a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})+\delta\ge a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})$. Therefore, we must have

$$-\delta\le a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=\cdots=a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_j,\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\le 0$$

For the expediting quantity, following properties can be derived based on the similar argument shown above. We only expedite inventory at leadtime position i if we have expedited all inventory at lower leadtime positions, which implies that $o^*_{i,t}(\bar{\boldsymbol{w}}-\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})=o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_k)$ for $k=1,\cdots,i$. Hence,

$$0\ge o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=\cdots=o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_i)-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}).$$

Since $o^*_{i,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})=R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})-\bar{v}_{i-1}$ and $R_{ij,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{v}})$ is nondecreasing in $(\bar{z}_n,\cdots,\bar{z}_j,\bar{v}_{i-1},\cdots,\bar{v}_l)$ and are independent of $(\bar{z}_{j-1},\cdots,\bar{z}_0,\bar{v}_0,\cdots,\bar{v}_{i-2})$, by a similar argument shown in the proof related to the optimal allocation policy, we can show that

$$\begin{cases}\delta\ge o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_{i+1})-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge\cdots\ge o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_{l+1})-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0,\\ \delta\ge o^*_{i,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge\cdots\ge o^*_{i,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{n+1},\bar{\boldsymbol{x}})-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0,\\ 0\ge o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=\cdots=o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_i)-o^*_{i,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge-\delta.\end{cases}$$

■

Proof of Theorem 3.

The optimal rationing level is independent of the on-hand inventory state $\hat{x}_0$. When making the allocation decisions for the demands, it is optimal to fulfill class j demand only if we have fulfilled all the demands of classes $0,\cdots,j-1$ (see in Lemma 2). Hence, the optimal rationing level of class j demand is also independent of $(\bar{w}_0,\cdots,\bar{w}_{j-1})$.

Recall that we have shown that $\bar{f}_t$ is L$^\natural$-convexity, which implies the convexity as indicated by Zipkins (2008). We thus define the optimal rationing level for class j demand, denoted by $r_{j,t}(\hat{\boldsymbol{x}}_{-0},\bar{w}_j,\cdots,\bar{w}_n)$, as the minimum r_j such that for arbitrarily small $\delta>0$

$$\begin{aligned}&b_j\delta+h\delta+\beta\bar{f}_{t+1}(\underbrace{-\sum_{k=j}^n\bar{w}_k-\delta,\cdots,-\sum_{k=j}^n\bar{w}_k-\delta}_{j+1},-\sum_{k=j+1}^n\bar{w}_k,\cdots,-\sum_{k=n}^n\bar{w}_k,-\sum_{k=j}^n\bar{w}_k+r_j+\sum_{k=1}^1\hat{x}_k,\cdots,\\&-\sum_{k=j}^n\bar{w}_k+r_j+\sum_{k=1}^l\hat{x}_k)]-\beta f_{t+1}(\underbrace{-\sum_{k=j}^n\bar{w}_k,\cdots,-\sum_{k=j}^n\bar{w}_k}_{j+1},-\sum_{k=j+1}^n\bar{w}_k,\cdots,-\sum_{k=n}^n\bar{w}_k,\\&-\sum_{k=j}^n\bar{w}_k+r_j+\sum_{k=1}^1\hat{x}_k,\cdots,-\sum_{k=j}^n\bar{w}_k+r_j+\sum_{k=1}^l\hat{x}_k)\ge 0,\end{aligned}$$

i.e., it is the minimum value under which the system is better off by fulfilling a small enough amount of class j demand. Due to the convexity of $\bar{f}_{t+1}$, we only allocate inventory to class j if $\hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k \geq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$.

We shall then show that $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ is independent of $(\bar{w}_j, \cdots, \bar{w}_n)$ as well. To accomplish this, we first show that $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \cdots, \bar{w}_n) = r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ for any $\delta > 0$. In other words, we shall show that the states $(0, \cdots, 0, \bar{w}_j, \cdots, \bar{w}_n, \hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$ and $(0, \cdots, 0, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n, \hat{x}_0 + \delta - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$ have the same optimal rationing level for demand class j, since the optimal rationing levels for the two states are $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ and $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n)$ respectively.

We let w_{m+} and $\hat{w}_{m+}$, $m \in \{0, \cdots, n\}$, be the remaining backorder quantities of class m under the two states $(0, \cdots, 0, \bar{w}_j, \cdots, \bar{w}_n, \hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$ and $(0, \cdots, 0, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n, \hat{x}_0 + \delta - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$, respectively. Under the case of $\hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k - r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ (otherwise the allocation decisions are not required for demands of classes $j, \cdots, n$),

$$\begin{cases} w_{j+} = \left[\sum_{k=0}^{j} \bar{w}_k - \hat{x}_0 + r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)\right]^+, \\ \hat{w}_{j+} = \left[\sum_{k=0}^{j} \bar{w}_k + \delta - \hat{x}_0 - \delta + r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n)\right]^+. \end{cases}$$

As shown in Theorem 2, under the same inventory state $\boldsymbol{x}$, increasing the backorder quantity of class j leads to an increase of the optimal allocation quantity, or equivalently, a decrease of the optimal reservation level for class j. Hence, $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \cdots, \bar{w}_n) \leq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$. Theorem 2 also implies that $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n) = r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$. $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n) \leq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_{j+k-1}, \bar{w}_{j+k} + \delta, \bar{w}_{j+k+1}, \cdots, \bar{w}_n) \leq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ for $k = 1, \cdots, n-j$, since the property $a^*_{j,t}(\boldsymbol{w} + \delta \boldsymbol{e}_{j+k}, \boldsymbol{x}) \leq a^*_{j,t}(\boldsymbol{w} + \delta \boldsymbol{e}_j, \boldsymbol{x}) \leq a^*_{j,t}(\boldsymbol{w}, \boldsymbol{x})$ holds for $k \geq 1$ and $\delta > 0$. We thus have $\hat{w}_{j+} \leq w_{j+}$, which implies that it must be optimal to fulfill at least δ amount of class j demand under the state $(0, \cdots, 0, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n, \hat{x}_0 + \delta - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$.

The allocation policy for class j can then be divided into two steps. First, we fulfill δ amount of class j demand under the state $(0, \cdots, 0, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n, \hat{x}_0 + \delta - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$ since in this case we know that it is optimal to fulfill at least δ amount of class j demand, and afterwards the resulting state is $(0, \cdots, 0, \bar{w}_j, \cdots, \bar{w}_n, \hat{x}_0 - \sum_{k=0}^{j-1} \bar{w}_k, \hat{\boldsymbol{x}}_{-0})$. It is clear that $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$ is the optimal rationing level under the resulting state after the first step. Therefore, $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n) = r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$.

Next, we show that the preceding result implies that the optimal rationing level $r_{j,t}$ are independent of all other backorder quantities. Recall that $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j + \delta, \bar{w}_{j+1}, \cdots, \bar{w}_n) \leq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \bar{w}_{j+1} + \delta, \bar{w}_{j+2}, \cdots, \bar{w}_n) \leq \cdots \leq r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$. It follows that

$$r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_{j+k-1}, \bar{w}_{j+k} + \delta, \bar{w}_{j+k+1}, \cdots, \bar{w}_n) = r_{j,t}(\hat{\boldsymbol{x}}_{-0}, \bar{w}_j, \cdots, \bar{w}_n)$$

for any $k = 1, \cdots, n-j$ and $\delta > 0$. We thus conclude that the optimal rationing level for class j is independent of the backorder quantities of stages $j, \cdots, n$ (i.e., $\bar{w}_j, \cdots, \bar{w}_n$).

In summary, (i) the optimal rationing levels for different stages are independent of the backorder quantities of different demand classes, and (ii) in order to determine the optimal rationing levels we just need to consider $r_{j,t}(\hat{\boldsymbol{x}}_{-0}, 0, \cdots, 0)$. ■

Proof of Lemma 4

The proof can be shown by induction. Note that $\bar{f}_{T+l+1}$ clearly is submodular in $\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}$. Suppose that $\bar{f}_{t+1}$ is submodular in $\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}$, then it is sufficient to show that $\bar{f}_t$ and $\bar{g}_t$ are submodular in $\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}$.

Note that

$$\bar{f}_t(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}) = \min_{v_l \geq v_{l-1}} \mathbb{E}\left[c(v_l - v_{l-1}) + \bar{g}_t\left(\tilde{\boldsymbol{b}}, z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, (\boldsymbol{v}, v_l) - \sum_{k=0}^{n} D_k \boldsymbol{e}\right)\right],$$

and $\bar{g}_t(\tilde{\boldsymbol{b}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = \min_{j=0,\cdots,n,\, i=1,\cdots,l} F_{ij,t}(\tilde{\boldsymbol{b}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$, where,

$$F_{ij,t}(\tilde{\boldsymbol{b}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = \min_{\bar{z}_j \leq y_j \leq \min\{\bar{z}_{j+1}, t_i\}, \bar{v}_{i-1} \leq t_i \leq \bar{v}_i} \left[\sum_{k=0}^{i-1} \hat{s}_k \bar{v}_k + s_i t_i - \sum_{k=j+1}^{n} (-\hat{b}_k)\bar{z}_k - \sum_{k=0}^{j} (-\hat{b}_k) y_j + h(t_i - y_j) \right.$$
$$\left. \beta \bar{f}_{t+1}(\tilde{\boldsymbol{b}}, \bar{z}_n, \cdots, \bar{z}_{j+1}, \overbrace{y_j, \cdots, y_j}^{j+1}, \overbrace{t_i, \cdots, t_i}^{i-1}, \bar{v}_i, \cdots, \bar{v}_l)\right],$$

for $j = 0, \cdots, n$ and $i = 1, \cdots, l$. Clearly, the function $F_{ij,t}$ here has the same properties with the function $F_{ij,t}$ in the proof of Theorem 1. Then, with the same reasons stated in the proof of Theorem 1 and also the fact that $-\sum_{k=j+1}^{n}(-\hat{b}_k)\bar{z}_k - \sum_{k=0}^{j}(-\hat{b}_k)y_j$ is submodular in $-\hat{b}_0, \cdots, -\hat{b}_n, \bar{z}_{j+1}, \cdots, \bar{z}_n, y_j$, we have that $\bar{g}_t$ is submodular in $(\tilde{\boldsymbol{b}}, \bar{\boldsymbol{z}}, \bar{\boldsymbol{v}})$. Note that the presence of $\tilde{\boldsymbol{b}}$ would not change the submodularity since $\hat{\boldsymbol{b}}$ are not decision variables. Since $v_l \geq v_{l-1}$ is a sublattice, from Theorem 2.7.6 of Topkis (1998) we know that submodularity can be preserved under the minimization over a sublattice. As a result, $\bar{f}_t$ is submodular in $(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v})$. That completes the inductive proof.

With the above result, we have that $q_t^*(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}) = S_t(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}) - v_{l-1}$ must be nondecreasing in $\tilde{\boldsymbol{b}}$. Hence, $q_t^*(\tilde{\boldsymbol{b}}, \boldsymbol{z}, \boldsymbol{v}) \leq q_t^*(b_0, 0, \cdots, 0, \boldsymbol{z}, \boldsymbol{v})$ since $\tilde{\boldsymbol{b}} \leq (b_0, 0, \cdots, 0,)$. ■

Proof of Proposition 2.

We show the results by induction. In the terminal period $T+l+1$, $\bar{f}_{T+l+1}(\boldsymbol{v})$ is decomposable as $\bar{f}_{T+l+1}(\boldsymbol{v}) \equiv 0$. Suppose that $\bar{f}_{t+1}(\boldsymbol{v})$ is decomposable and can be expressed as $\bar{f}_{t+1}(\boldsymbol{v}) = \sum_{i=0}^{l-1} \bar{f}_{i,t+1}(v_i)$ for the convex functions $\bar{f}_{i,t+1}(\cdot)$'s. We shall show that $\bar{f}_t(\boldsymbol{v})$ is decomposable as well.

Based on the decomposition of $\bar{f}_{t+1}(\boldsymbol{v})$, we have

$$\bar{g}_t(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = -s_1(\bar{v}_0 - \bar{v}_0 \vee 0) + \sum_{i=1}^{l}(s_i - s_{i+1})(\bar{v}_i - \bar{v}_i \vee 0) + h(\bar{v}_0 \vee 0) + \beta \sum_{i=1}^{l} \bar{f}_{i-1,t+1}(\bar{v}_i \vee 0).$$

Let

$$\begin{cases} \bar{g}_{0,t}(\zeta) = -s_1(\zeta - \zeta \vee 0) + h(\zeta \vee 0), \\ \bar{g}_{1,t}(\zeta) = (s_1 - s_2)(\zeta - \zeta \vee 0) + \beta \bar{f}_{0,t+1}(\zeta \vee 0), \\ \bar{g}_{i,t}(\zeta) = (s_i - s_{i+1})(\zeta - \zeta \vee 0) + \beta \bar{f}_{i-1,t+1}(\zeta \vee 0), \quad i = 2, \cdots, l. \end{cases}$$

Then, $\bar{g}_t(\bar{\boldsymbol{z}}, \bar{\boldsymbol{v}}) = \sum_{i=0}^{l} \bar{g}_{i,t}(\bar{v}_i)$, where $\bar{g}_{i,t}(\cdot)$'s are all single-variable convex functions. Let

$$\mathbb{E}[c(v_l - v_{l-1}) + \bar{g}_t((\boldsymbol{v}, v_l) - (D_0 + D_1)\boldsymbol{e})] = \sum_{i=0}^{l} G_{i,t}(v_i),$$

where

$$\begin{cases} G_{i,t}(\zeta) = \mathbb{E}[\bar{g}_{i,t}(\zeta - D_0 - D_1)], & i = 0, \cdots, l-2, \\ G_{l-1,t}(\zeta) = -c\zeta + \mathbb{E}[\bar{g}_{l-1,t}(\zeta - D_0 - D_1)], & \\ G_{l,t}(\zeta) = c\zeta + \mathbb{E}[\bar{g}_{l,t}(\zeta - D_0 - D_1)]. & \end{cases}$$

Now define

$$S_t = \min \arg\min_{\zeta} G_{l,t}(\zeta).$$

The constant S_t is the base stock level. Based on convexity, the optimal ordering decision in period t is as follows: if $v_{l-1} < S_t$, then it is optimal to order up to S_t; otherwise, it is optimal to ordering nothing. Accordingly, under the optimal policy, $v_l^* = v_{l-1} \vee S_t$ and the function $G_{l,t}(v_l^*) = G_{l,t}(v_{l-1} \vee S_t)$.

Finally, we let

$$\begin{cases} \bar{f}_{i,t}(\zeta) = G_{i,t}(\zeta), & i = 0, \cdots, l-2, \\ \bar{f}_{l-1,t}(\zeta) = G_{l-1,t}(\zeta) + G_{l,t}(\zeta \vee S_t). \end{cases}$$

Then, the value function $\bar{f}_t(\boldsymbol{v})$ is decomposable and can be expressed as

$$\bar{f}_t(\boldsymbol{v}) = \sum_{i=0}^{l-1} \bar{f}_{i,t}(v_i)$$

with the convex functions $\bar{f}_{i,t}(\zeta)$'s.

We thus conclude that a base stock level is optimal for the ordering decision and the value function $\bar{f}_t(\boldsymbol{v})$ is decomposable in each period t. ■

Proof of Proposition 3.
Let $D = D_0 + D_1$. Then, based on (14) - (16),

$$\begin{aligned} \bar{f}_{0,t}(\zeta) =& \mathbb{E}\left[-s_1(\zeta - D - (\zeta - D) \vee 0) + h((\zeta - D) \vee 0)\right], \\ \bar{f}_{i,t}(\zeta) =& \mathbb{E}\left[(s_i - s_{i+1})(\zeta - D - (\zeta - D) \vee 0) + \beta \bar{f}_{i-1,t+1}((\zeta - D) \vee 0)\right], \quad i = 1, \cdots, l-2, \\ \bar{f}_{l-1,t}(\zeta) =& \mathbb{E}\left[-c\zeta + (s_{l-1} - s_l)(\zeta - D - (\zeta - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((\zeta - D) \vee 0)\right] \\ &+ \mathbb{E}\left[c(\zeta \vee S_t) + (s_l - s_{l-1})(\zeta \vee S_t - D - (\zeta \vee S_t - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((\zeta \vee S_t - D) \vee 0)\right], \end{aligned}$$

where S_t is defined as in Proposition 2. The expression of $\bar{f}_{0,t}(\zeta)$ is explicitly given for any t. Then, $\bar{f}_{1,t}(\zeta)$ can also be obtained explicitly based on the above formula. Based on this logic, we can recursively obtain the explicit expressions for $\bar{f}_{2,t}(\zeta), \cdots, \bar{f}_{l-2,t}(\zeta)$.

Recall that $S_t = \min \arg\min_{\zeta} \left[(c + s_l - s_{l+1})\zeta + \mathbb{E}[(s_{l+1} - s_l)((\zeta - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((\zeta - D) \vee 0)\right]$. Then, the function $\bar{f}_{l-1,t}(v_{l-1,t})$ can be equivalently expressed as

$$\begin{aligned} \bar{f}_{l-1,t}(v_{l-1,t}) =& \mathbb{E}\left[-cv_{l-1,t} + (s_{l-1} - s_l)(v_{l-1,t} - D - (v_{l-1,t} - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((v_{l-1,t} - D) \vee 0)\right] \\ &+ (s_{l+1} - s_{l-1})(v_{l-1,t} \vee S_t) - \mathbb{E}\left[(s_l - s_{l-1})D\right] + \mathbb{E}\left[(s_{l-1} - s_{l+1})((v_{l-1,t} \vee S_t - D) \vee 0)\right] \\ &+ \mathbb{E}\left[(c + s_l - s_{l+1})(v_{l-1,t} \vee S_t) + (s_{l+1} - s_l)((v_{l-1,t} \vee S_t - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((v_{l-1,t} \vee S_t - D) \vee 0)\right] \\ =& \mathbb{E}\left[-cv_{l-1,t} + (s_{l-1} - s_l)(v_{l-1,t} - (v_{l-1,t} - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((v_{l-1,t} - D) \vee 0)\right] \\ &+ (s_{l+1} - s_{l-1})(v_{l-1,t} \vee S_t) + \mathbb{E}\left[(s_{l-1} - s_{l+1})((v_{l-1,t} \vee S_t - D) \vee 0)\right] \\ &+ \min_{v_{l,t} \geq v_{l-1,t}} \mathbb{E}\left[(c + s_l - s_{l+1})v_{l,t} + (s_{l+1} - s_l)((v_{l,t} - D) \vee 0) + \beta \bar{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)\right]. \end{aligned}$$

Define

$$U_t(v_{l-1,t}) = \mathbb{E}\left[-cv_{l-1,t} + (s_{l-1} - s_l)(v_{l-1,t} - (v_{l-1,t} - D) \vee 0) + \beta \bar{f}_{l-2,t+1}((v_{l-1,t} - D) \vee 0)\right]$$

$$+ (s_{l+1} - s_{l-1})(v_{l-1,t} \vee S_t) + \mathbb{E}\left[(s_{l-1} - s_{l+1})((v_{l-1,t} \vee S_t - D) \vee 0)\right],$$
$$V_t(v_{l,t}) = \mathbb{E}\left[(c + s_l - s_{l+1})v_{l,t} + (s_{l+1} - s_l)((v_{l,t} - D) \vee 0)\right],$$

where $V_t(v_{l,t})$ is convex in $v_{l,t}$. Then, $\bar{f}_{l-1,t}(v_{l-1,t}) = U_t(v_{l-1,t}) + \min_{v_{l,t} \geq v_{l-1,t}} \left(V_t(v_{l,t}) + \beta \bar{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)\right)$, where the objective function $V_t(v_{l,t}) + \beta \bar{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)$ is convex. Further define

$$\tilde{f}_{l-1,t}(v_{l-1,t}) = \bar{f}_{l-1,t}(v_{l-1,t}) - U_t(v_{l-1,t}), \quad t = 1, \cdots, T + l + 1,$$

which is convex based on Proposition B-4 in Heyman and Sobel (1984). Then, for $t = 1, \cdots, T + l$, we have

$$\tilde{f}_{l-1,t}(v_{l-1,t}) = \min_{v_{l,t} \geq v_{l-1,t}} \left(g_t(v_{l,t}) + \beta \mathbb{E}\left[\tilde{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)\right]\right)$$

where $g_t(v_{l,t}) = V_t(v_{l,t}) + \beta \mathbb{E}\left[U_{t+1}((v_{l,t} - D) \vee 0)\right]$. Accordingly, S_t can be equivalently obtained by

$$S_t = \min \arg\min_{\zeta} \left(g_t(\zeta) + \beta \mathbb{E}\left[\tilde{f}_{l-1,t+1}((\zeta - D) \vee 0)\right]\right).$$

We also assume that the terminal condition of $\tilde{f}_{l-1,T+l+1}(\zeta) \equiv 0$ for any ζ. Note that the expression of $g_t(v_{l,t})$ is explicitly given and hence

$$\tilde{S} = \min \arg\min_{\zeta} g_t(\zeta)$$

is independent of t, i.e., $\tilde{S}$ is the global minimizer of $g_t(\zeta)$ for any $t = 1, \cdots, T + l$.

We then show the optimality of the myopic ordering policy by induction. Let $H_t(v_{l,t}) = g_t(v_{l,t}) + \beta \mathbb{E}\left[\tilde{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)\right]$. Then, $\tilde{f}_{l-1,t}(v_{l-1,t}) = \min_{v_{l,t} \geq v_{l-1,t}} H_t(v_{l,t})$. In period $T + l$, $S_{T+l} = \tilde{S}$. If $v_{l-1,T+l} \leq \tilde{S}$, we have $\tilde{f}_{l-1,T+l}(v_{l-1,T+l}) = g_{T+l}(\tilde{S})$, i.e., the myopic base stock level $\tilde{S}$ is optimal in period $T + l$. Due to the convexity, $\tilde{f}_{l-1,T+l}(v_{l-1,T+l}) = g_{T+l}(v_{l-1,T+l} \vee \tilde{S})$, which is nondecreasing in $v_{l-1,T+l}$. Hence, the derivative of $\tilde{f}_{l-1,T+l}(v_{l-1,T+l})$ is

$$\tilde{f}'_{l-1,T+l}(v_{l-1,T+l}) = \begin{cases} 0, & \text{if } v_{l-1,T+l} \leq \tilde{S}, \\ H'_{T+l}(v_{l-1,T+l}) \geq 0, & \text{otherwise.} \end{cases}$$

Now suppose that in period $t + 1$, $t = 1, \cdots, T + l - 1$, $\tilde{f}'_{l-1,t+1}(v_{l-1,t+1}) = 0$ if $v_{l-1,t+1} \leq \tilde{S}$ and $\tilde{f}'_{l-1,t+1}(v_{l-1,t+1}) = H'_{t+1}(v_{l-1,t+1}) \geq 0$ otherwise. It suffices to show that $\tilde{f}_{l-1,t}(v_{l-1,t})$ has the same property. Recall the definitions of S_t and $\tilde{S}$ above. As $\tilde{f}_{l-1,t+1}(v_{l-1,t+1})$ is nondecreasing, we must have $S_t \leq \tilde{S}$. Then, $S_t - D \leq \tilde{S}$ and hence $\tilde{f}'_{l-1,t+1}((S_t - D) \vee 0) = 0$ must hold based on the inductive assumption. This implies that if $v_{l-1,t} \leq \tilde{S}$ in period t, we must have $S_t = \tilde{S}$ and

$$\tilde{f}_{l-1,t}(v_{l-1,t}) = \min_{v_{l,t} \geq v_{l-1,t}} \left(g_t(v_{l,t}) + \beta \mathbb{E}\left[\tilde{f}_{l-1,t+1}((v_{l,t} - D) \vee 0)\right]\right) = H_t(\tilde{S}).$$

If $v_{l-1,t} > \tilde{S}$ in period t, then $\tilde{f}_{l-1,t}(v_{l-1,t}) = H_t(v_{l-1,t})$. As $H_t(v_{l-1,t})$ is convex, we have

$$\tilde{f}'_{l-1,t}(v_{l-1}) = \begin{cases} 0, & \text{if } v_{l-1} \leq \tilde{S}, \\ H'_t(v_{l-1}) \geq 0, & \text{otherwise.} \end{cases}$$

We thus have completed the proof and show that a myopic ordering policy with the base stock level $\tilde{S}$ is optimal.

Therefore, if the initial inventory level $v_{l-1,1} \le \tilde{S}$ in period 1, we can guarantee that under the optimal policy $v_{l-1,t} \le \tilde{S}$ for every period $t = 1, \cdots, T+l$ and hence a myopic ordering policy with the base stock level $\tilde{S}$ defined above is optimal for the whole planning horizon. ■

Explanation of The Expediting Costs for the System with Partially Full Expediting.
We explain the total expediting cost in (19) as follows. Based on our definitions of v, $\bar{v}$ and $u_{i,t}$'s, $v \vee u_{i,t}$ and $\bar{v} \vee u_{i,t}$ are the negatively aggregated inventory level (NAIL) of leadtime position i before and after the fulfillment of class 0 demands at period t, respectively, while $v \vee u_{i+1,t}$ represents the NAIL of leadtime position $i+1$ before the fulfillment of class 0 demands. Then, $v \vee u_{i+1,t} - v \vee u_{i,t}$ is the quantity of the inventory at leadtime position i at the beginning of period t, while $\bar{v} \vee u_{i,t} - v \vee u_{i,t}$ indicates the quantity of the inventory that we expedite from leadtime poistions $i, i+1, \cdots, l+1$ to fulfill class 0 demands in period t. Hence, due to the sequentially expediting property, $(\bar{v} \vee u_{i,t}) \wedge (v \vee u_{i+1,t}) - v \vee u_{i,t}$ is the quantity of the inventory expedited from leadtime position i. For leadtime position $l+1$, it has a sufficiently large stock and hence $\bar{v} \vee u_{l+1,t} - v \vee u_{l+1,t}$ is the expedited inventory from leadtime position $l+1$ in order to fulfill class 0 demands.

Note that $a \vee (x \wedge b) = a \vee x + b - b \vee x$ and $(x \vee a) \wedge (y \vee b) = y \vee (b \wedge x) + a \vee (b \wedge x) - b \wedge x$ for $x \ge y$ and $a \le b$. Through some algebra, we have $(\bar{v} \vee u_{i,t}) \wedge (v \vee u_{i+1,t}) = v \vee u_{i+1,t} - \bar{v} \vee u_{i+1,t} + u_{i,t} \vee \bar{v}$. Hence, (19) can be equivalently written as (20).

Proof of Proposition 4.
We show the results by induction. In period $T+l+1$, the result must hold. Suppose that $\bar{g}_{t+1}(z, v \vee \boldsymbol{u}_{t+1})$ is decomposable, we shall show that the property holds in period t as well.

When $\bar{g}_{t+1}(z, v \vee \boldsymbol{u}_{t+1})$ is decomposable, we have

$$\bar{g}_{t+1}(\bar{z}, y \vee \boldsymbol{u}_{t+1}) = \tilde{g}_{t+1}(\bar{z}) + \bar{g}_{0,t+1}(y) + \sum_{i=1}^{l+1} \bar{g}_{i,t+1}(y \vee u_{i,t+1})$$

as $\bar{v} \ge u_{0,t}$ in all periods $1, \cdots, T+l+1$. Let

$$R_t = \min \arg\min_{y} \left[s_1(y \vee u_{1,t}) + \sum_{i=2}^{l+1} (s_i - s_{i-1})(y \vee u_{i,t}) + \beta \bar{g}_{0,t+1}(y) + \beta \sum_{i=1}^{l+1} \bar{g}_{i,t+1}(y \vee u_{i,t+1}) \right].$$

Then, the optimal solution of y in (21), denoted by y^*, shall be $y^* = (\bar{v} \vee R_t) \wedge \bar{z}$. With (18), $\hat{g}_t(\bar{z}, \bar{v} \vee \boldsymbol{u}_t)$ can be expressed as follows:

$$\begin{aligned}
&\hat{g}_t(\bar{z}, \bar{v} \vee \boldsymbol{u}_t) \\
=& s_1(y^* \vee u_{1,t}) + \sum_{i=2}^{l+1} (s_i - s_{i-1})(y^* \vee u_{i,t}) + \beta \bar{g}_{t+1}(\bar{z}, y^* \vee \boldsymbol{u}_{t+1}) \\
=& s_1(u_{1,t} \vee \bar{z} + u_{1,t} \vee \bar{v} \vee R_t - u_{1,t} \vee R_t \vee \bar{z}) + \sum_{i=2}^{l+1} (s_i - s_{i-1})(u_{i,t} \vee \bar{z} + u_{i,t} \vee R_t \vee \bar{v} \\
& - u_{i,t} \vee R_t \vee \bar{z}) + \beta \tilde{g}_{t+1}(\bar{z}) + \beta \bar{g}_{0,t+1}(\bar{z}) + \beta \sum_{i=1}^{l} \bar{g}_{i,t+1}(u_{i+1,t} \vee \bar{z}) + \beta \bar{g}_{l+1,t+1}(u_{l+1,t} \vee \bar{z}) \\
& + \beta \bar{g}_{0,t+1}(R_t \vee \bar{v}) + \beta \sum_{i=1}^{l} \bar{g}_{i,t+1}(u_{i+1,t} \vee R_t \vee \bar{v}) + \beta \bar{g}_{l+1,t+1}(u_{l+1,t} \vee R_t \vee \bar{v})
\end{aligned}$$

$$-\beta\bar{g}_{0,t+1}(R_t\vee\bar{z})-\beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t}\vee R_t\vee\bar{z})-\beta\bar{g}_{l+1,t+1}(u_{l+1,t}\vee R_t\vee\bar{z}).$$

Define

$$\begin{aligned}\hat{G}_{1,t}(\bar{z})=&s_1(u_{1,t}\vee\bar{z}-u_{1,t}\vee R_t\vee\bar{z})+\sum_{i=2}^{l+1}(s_i-s_{i-1})(u_{i,t}\vee\bar{z}-u_{i,t}\vee R_t\vee\bar{z})+\beta\tilde{g}_{t+1}(\bar{z})\\&+\beta\bar{g}_{0,t+1}(\bar{z})+\beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t}\vee\bar{z})+\beta\bar{g}_{l+1,t+1}(u_{l+1,t}\vee\bar{z})\\&-\beta\bar{g}_{0,t+1}(R_t\vee\bar{z})-\beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t}\vee R_t\vee\bar{z})-\beta\bar{g}_{l+1,t+1}(u_{l+1,t}\vee R_t\vee\bar{z}),\\\hat{G}_{2,t}(\bar{v})=&s_1(u_{1,t}\vee R_t\vee\bar{v})+\sum_{i=2}^{l+1}(s_i-s_{i-1})(u_{i,t}\vee R_t\vee\bar{v})\\&+\beta\bar{g}_{0,t+1}(R_t\vee\bar{v})+\beta\sum_{i=1}^{l}\bar{g}_{i,t+1}(u_{i+1,t}\vee R_t\vee\bar{v})+\beta\bar{g}_{l+1,t+1}(u_{l+1,t}\vee R_t\vee\bar{v})\end{aligned}$$

so that $\hat{g}_t(\bar{z},\bar{v}\vee\boldsymbol{u}_t)=\hat{G}_{1,t}(\bar{z})+\hat{G}_{2,t}(\bar{v})$. Further define

$$\begin{aligned}\bar{g}_{1,t}(v\vee u_{1,t})=&-s_1(v\vee u_{1,t}),\\\bar{g}_{i,t}(v\vee u_{i,t})=&-(s_i-s_{i-1})(v\vee u_{i,t}),\qquad\qquad i=1,\cdots,l+1.\end{aligned}$$

Then,

$$\bar{g}_t(z,v\vee\boldsymbol{u}_t)=\tilde{g}_t(z)+\bar{g}_{0,t}(v)+\sum_{i=1}^{l+1}\bar{g}_{i,t}(v\vee u_{i,t}),$$

where $\bar{g}_{i,t}(\zeta)$ for $i=1,\cdots,l+1$ are defined above and

$$\begin{cases}\tilde{g}_t(z)=\mathbb{E}\left[\hat{G}_{1,t}(z+D_0+D_1)\right],\\\bar{g}_{0,t}(v)=\mathbb{E}\left[\hat{G}_{2,t}(v+D_0)\right].\end{cases}$$

■

Proof of Proposition 5.

First, we show the convexity of $\bar{f}_t(\boldsymbol{v})$ and $\bar{g}_t(\bar{\boldsymbol{v}})$. $\bar{f}_{T+l+1}(\boldsymbol{v})$ is clearly convex. Suppose that $\bar{f}_{t+1}(\boldsymbol{v})$ is convex, we shall show that $\bar{g}_t(\bar{\boldsymbol{v}})$ and $\bar{f}_t(\boldsymbol{v})$ are convex. In the following, we shall use the conclusion (3.10) on page 84 in Boyd and Vandenberghe (2009), i.e., if $f:\mathbb{R}^n\to\mathbb{R}$ is convex and nondecreasing, then $f(x\vee y)$ is convex in (x,y); and Proposition B-4 in Heyman and Sobel (1984), i.e., if $f:\mathbb{R}^n\to\mathbb{R}$ is a convex function, then $g(\boldsymbol{x})=\min_{y\in A(\boldsymbol{x})}f(y,\boldsymbol{x})$ must be convex as well if $A(\boldsymbol{x})$ is a convex set.

Let

$$\begin{aligned}G_t(y,\bar{\boldsymbol{v}})=&h(y\vee 0)+b_0(y\vee 0-y)+\sum_{i=1}^{l}s_i(\bar{v}_i-y\vee\bar{v}_i-\bar{v}_{i-1}+y\vee\bar{v}_{i-1})+\beta\bar{f}_{t+1}(y\vee\bar{v}_1,\cdots,y\vee\bar{v}_l)\\=&(h+b_0)(y\vee 0)-b_0y-s_1(\bar{v}_0-y)+\sum_{i=1}^{l-1}(s_i-s_{i+1})(\bar{v}_i-y\vee\bar{v}_i)+\beta\bar{f}_{t+1}(y\vee\bar{v}_1,\cdots,y\vee\bar{v}_l).\end{aligned}$$

Then, $\bar{g}_t(\bar{\boldsymbol{v}})=\min_{\bar{v}_0\le y\le\bar{v}_l}G_t(y,\bar{\boldsymbol{v}})$. Due to Lemma 3 and the convexity of $\bar{f}_{t+1}(\boldsymbol{v})$, $\beta\bar{f}_{t+1}(y\vee\bar{v}_1,\cdots,y\vee\bar{v}_l)+(s_{i+1}-s_i)(y\vee\bar{v}_i)$ is convex and nondecreasing in $(y\vee\bar{v}_i)$. Moreover, $(h+b_0)(y\vee 0)$ is also convex and nondecreasing in y. Hence, $G_t(y,\bar{\boldsymbol{v}})$ is convex in $(y,\bar{\boldsymbol{v}})$. Further, due to Proposition B-4 in Heyman and Sobel (1984), we conclude that $\bar{g}_t(\bar{\boldsymbol{v}})$ is convex in $\bar{\boldsymbol{v}}$ and accordingly $\bar{f}_t(\boldsymbol{v})$ is convex in $\boldsymbol{v}$. Hence, the convexity of each component function of $\bar{f}_t(\boldsymbol{v})$ and $\bar{g}_t(\bar{\boldsymbol{v}})$ must hold. We thus omit the proof for convexity in the following.

Next, we show the decomposition of $\bar{f}_t(\boldsymbol{v})$. In the terminal period $T+l+1$, $\bar{f}_{T+l+1}(\boldsymbol{v})$ is clearly decomposable. Suppose that the decomposition holds in period $t+1$, we shall show that it holds in period t as well.

Based on the decomposition of $\bar{f}_{t+1}(\boldsymbol{v})$, i.e., $\bar{f}_{t+1}(\boldsymbol{v})=\sum_{i=0}^{l-1}\bar{f}_{i,t+1}(v_i)$,

$$G_t(y,\bar{\boldsymbol{v}})=(h+b_0)(y\vee 0)+(s_1-b_0)y-s_1\bar{v}_0+\sum_{i=1}^{l-1}(s_i-s_{i+1})(\bar{v}_i-y\vee\bar{v}_i)+\beta\sum_{i=1}^{l}\bar{f}_{i-1,t+1}(y\vee\bar{v}_i),$$

and $\bar{g}_t(\bar{\boldsymbol{v}})=\min_{\bar{v}_0\le y\le\bar{v}_l}G_t(y,\bar{\boldsymbol{v}})$.

Define

$$\Omega_{i,t}(y)=\begin{cases}(h+b_0)(y\vee 0)+(s_1-b_0)y, & i=1,\\ (h+b_0)(y\vee 0)+(s_1-b_0)y-\sum_{k=1}^{i-1}(s_k-s_{k+1})y+\beta\sum_{k=1}^{i-1}\bar{f}_{k-1,t+1}(y), & i=2,\cdots,l.\end{cases}$$

Then, $\Omega_{i,t}(y)$ is the function collecting all the terms depending on y when $\bar{v}_{i-1}\le y\le\bar{v}_i$ for $i=1,\cdots,l$ in the expression of $G_t(y,\bar{\boldsymbol{v}})$. The function $\Omega_{i,t}(y)$ is convex in y due to the convexity of $\bar{f}_{k-1,t+1}(\cdot)$ for $k=1,\cdots,l-1$. Let

$$r_{i,t}=\min\arg\min_{y}\Omega_{i,t}(y),\quad i=1,\cdots,l.$$

We then shall show that $r_{1,t}\ge\cdots\ge r_{l,t}$. Note that

$$\Omega_{i+1,t}(y)-\Omega_{i,t}(y)=\begin{cases}(s_2-s_1)y+\beta\bar{f}_{0,t+1}(y), & i=1,\\ (s_{i+1}-s_i)y+\beta\bar{f}_{i-1,t+1}(y), & i=2,\cdots,l-1.\end{cases}$$

Based on Lemma 3, $\bar{f}_{t+1}(\boldsymbol{v})+\frac{\Theta'}{\beta}\boldsymbol{v}$ is nondecreasing in $v_0,\cdots,v_{l-2}$, where $\Theta=(s_2-s_1,\cdots,s_l-s_{l-1},0)$. Then, $\Omega_{i+1,t}(y)-\Omega_{i,t}(y)$ is nondecreasing in y and hence $r_{1,t}\ge\cdots\ge r_{l,t}$.

Recall that $r_{i,t}$ is indeed the global minimizer of the function $G_t(y,\bar{\boldsymbol{v}})$ when $\bar{v}_{i-1}\le y\le\bar{v}_i$. Hence, within the range $[\bar{v}_{i-1},\bar{v}_i]$, the minimizer of $G_t(y,\bar{\boldsymbol{v}})$ shall be $(\bar{v}_{i-1}\vee r_{i,t})\wedge\bar{v}_i$. As $r_{1,t}\ge\cdots\ge r_{l,t}$ and $\bar{v}_0\le\cdots\le\bar{v}_l$, the optimal value of y, denoted by R_t, for $\bar{v}_0\le y\le\bar{v}_l$ is given below:

$$R_t=\sum_{i=1}^{l}\mathbb{I}_{\{\bar{v}_{i-1}\le r_{i,t}<\bar{v}_i\}}r_{i,t}+\sum_{i=1}^{l-1}\mathbb{I}_{\{r_{i+1,t}<\bar{v}_i\le r_{i,t}\}}\bar{v}_i+\mathbb{I}_{\{r_{1,t}<\bar{v}_0\}}\bar{v}_0+\mathbb{I}_{\{r_{l,t}\ge\bar{v}_l\}}\bar{v}_l,$$

and

$$G_t(R_t,\bar{\boldsymbol{v}})=\sum_{i=1}^{l}\mathbb{I}_{\{\bar{v}_{i-1}\le r_{i,t}<\bar{v}_i\}}G_t(r_{i,t},\bar{\boldsymbol{v}})+\sum_{i=1}^{l-1}\mathbb{I}_{\{r_{i+1,t}<\bar{v}_i\le r_{i,t}\}}G_t(\bar{v}_i,\bar{\boldsymbol{v}})+\mathbb{I}_{\{r_{1,t}<\bar{v}_0\}}G_t(\bar{v}_0,\bar{\boldsymbol{v}})+\mathbb{I}_{\{r_{l,t}\ge\bar{v}_l\}}G_t(\bar{v}_l,\bar{\boldsymbol{v}}).$$

As

$$\begin{cases}(\bar{v}_{i-1}\vee r_{i,t})\wedge\bar{v}_i=\mathbb{I}_{\{\bar{v}_{i-1}\le r_{i,t}<\bar{v}_i\}}r_{i,t}+\mathbb{I}_{\{r_{i,t}<\bar{v}_{i-1}\}}\bar{v}_{i-1}+\mathbb{I}_{\{r_{i,t}\ge\bar{v}_i\}}\bar{v}_i,\\ \bar{v}_i=\mathbb{I}_{\{r_{i+1,t}<\bar{v}_i\le r_{i,t}\}}\bar{v}_i+\mathbb{I}_{\{\bar{v}_i\le r_{i+1,t}\}}\bar{v}_i+\mathbb{I}_{\{\bar{v}_i>r_{i,t}\}}\bar{v}_i,\end{cases}$$

the expression of R_t can be equivalently expressed as

$$R_t=\sum_{i=1}^{l}(\bar{v}_{i-1}\vee r_{i,t})\wedge\bar{v}_i-\sum_{i=1}^{l-1}\bar{v}_i.$$

Accordingly,

$$\begin{aligned}\bar{g}_t(\bar{\boldsymbol{v}}) = G_t(R_t, \bar{\boldsymbol{v}}) &= \sum_{i=1}^{l} G_t((\bar{v}_{i-1} \vee r_{i,t}) \wedge \bar{v}_i, \bar{\boldsymbol{v}}) - \sum_{i=1}^{l-1} G_t(\bar{v}_i, \bar{\boldsymbol{v}}) \\ &= \sum_{i=0}^{l-1} G_t(\bar{v}_i \vee r_{i+1,t}, \bar{\boldsymbol{v}}) + G_t(\bar{v}_l, \bar{\boldsymbol{v}}) - \sum_{i=1}^{l} G_t(r_{i,t} \vee \bar{v}_i, \bar{\boldsymbol{v}})\end{aligned}$$

Since given any constant r,

$$\begin{aligned}G_t(r \vee \bar{v}_0, \bar{\boldsymbol{v}}) =& (h+b_0)(r \vee \bar{v}_0 \vee 0) + (s_1 - b_0)(r \vee \bar{v}_0) - s_1 \bar{v}_0 + \sum_{k=1}^{l-1}(s_k - s_{k+1})(\bar{v}_k - r \vee \bar{v}_k) \\ &+ \beta \sum_{k=1}^{l} \bar{f}_{k-1,t+1}(r \vee \bar{v}_k),\end{aligned}$$

and for $i = 1, \cdots, l$,

$$\begin{aligned}G_t(r \vee \bar{v}_i, \bar{\boldsymbol{v}}) =& (h+b_0)(r \vee \bar{v}_i \vee 0) + (s_1 - b_0)(r \vee \bar{v}_i) - s_1 \bar{v}_0 + \sum_{k=1}^{i}(s_k - s_{k+1})(\bar{v}_k - r \vee \bar{v}_i) \\ &+ \sum_{k=i+1}^{l-1}(s_k - s_{k+1})(\bar{v}_k - r \vee \bar{v}_k) + \beta \sum_{k=1}^{i} \bar{f}_{k-1,t+1}(r \vee \bar{v}_i) + \beta \sum_{k=i+1}^{l} \bar{f}_{k-1,t+1}(r \vee \bar{v}_k),\end{aligned}$$

the function $G_t(r \vee \bar{v}_i, \bar{\boldsymbol{v}})$, $i = 0, \cdots, l$, is indeed decomposable and can be expressed as the sum of single-variable convex functions. So is the function $\bar{g}_t(\bar{\boldsymbol{v}})$, i.e.,

$$\bar{g}_t(\bar{\boldsymbol{v}}) = \sum_{i=0}^{l} \bar{g}_{i,t}(\bar{v}_i),$$

where

$$\begin{aligned}\bar{g}_{0,t}(\bar{v}_0) =& (h+b_0)(r_{1,t} \vee \bar{v}_0 \vee 0) + (s_1 - b_0)(r_{1,t} \vee \bar{v}_0) - s_1 \bar{v}_0, \\ \bar{g}_{1,t}(\bar{v}_1) =& (h+b_0)(r_{2,t} \vee \bar{v}_1 \vee 0 - r_{1,t} \vee \bar{v}_1 \vee 0) + (s_2 - b_0)(r_{2,t} \vee \bar{v}_1 - r_{1,t} \vee \bar{v}_1) + (s_1 - s_2)(\bar{v}_1 - r_{1,t} \vee \bar{v}_1) \\ &+ \beta \bar{f}_{0,t+1}(r_{2,t} \vee \bar{v}_1) \\ \bar{g}_{i,t}(\bar{v}_i) =& (h+b_0)(r_{i+1,t} \vee \bar{v}_i \vee 0 - r_{i,t} \vee \bar{v}_i \vee 0) + (s_{i+1} - b_0)(r_{i+1,t} \vee \bar{v}_i - r_{i,t} \vee \bar{v}_i) + (s_i - s_{i+1})(\bar{v}_i - r_{1,t} \vee \bar{v}_i) \\ &+ \sum_{k=2}^{i}(s_i - s_{i+1})(r_{k-1,t} \vee \bar{v}_i - r_{k,t} \vee \bar{v}_i) + \beta \sum_{k=1}^{i} \bar{f}_{k-1,t+1}(r_{i+1,t} \vee \bar{v}_i) - \beta \sum_{k=1}^{i-1} \bar{f}_{k-1,t+1}(r_{i,t} \vee \bar{v}_i), \\ & i = 2, \cdots, l-1, \\ \bar{g}_{l,t}(\bar{v}_{l,t}) =& (h+b_0)(\bar{v}_l \vee 0 - r_{l,t} \vee \bar{v}_l \vee 0) + (s_{l+1} - b_0)(\bar{v}_l - r_{l,t} \vee \bar{v}_l) + \beta \sum_{k=1}^{l} \bar{f}_{k-1,t+1}(\bar{v}_l) - \beta \sum_{k=1}^{l-1} \bar{f}_{k-1,t+1}(r_{l,t} \vee \bar{v}_l).\end{aligned}$$

With the decomposition of $\bar{g}_t(\bar{\boldsymbol{v}})$,

$$\mathbb{E}[c(v_l - v_{l-1}) + \bar{g}_t((\boldsymbol{v}, v_l) - D\boldsymbol{e})] = \mathbb{E}\left[-cv_{l-1} + \sum_{i=0}^{l-1} \bar{g}_{i,t}(v_i - D)\right] + \mathbb{E}\left[cv_l + \bar{g}_{l,t}(v_l - D)\right].$$

Define

$$S_t = \min \arg\min \mathbb{E}\left[cv_l + \bar{g}_{l,t}(v_l - D)\right].$$

Then, the optimal ordering decision can be described by a base stock policy: It is optimal to order up to the fixed base stock level S_t when $v_{l-1} < S_t$ and order nothing otherwise. That is $v_l^* = v_{l-1} \vee S_t$. Then, we conclude that

$$\bar{f}_t(\boldsymbol{v}) = \sum_{i=0}^{l-1} \bar{f}_{i,t}(v_i),$$

where

$$\begin{cases} \bar{f}_{i,t}(\zeta) = \mathbb{E}[\bar{g}_{i,t}(\zeta - D)], & i = 0, \cdots, l-2, \\ \bar{f}_{l-1,t}(\zeta) = c(\zeta \vee S_t) - c\zeta + \mathbb{E}[\bar{g}_{l-1,t}(\zeta - D)] + \mathbb{E}[\bar{g}_{l,t}(\zeta \vee S_t - D)]. \end{cases}$$

The convexity of $\bar{f}_{i,t}(\cdot)$'s is guaranteed by the convexity of $\bar{f}_t(\boldsymbol{v})$ which we have shown before. We thus have completed the proof. ∎

Proof of Proposition 6.
As $\bar{f}_{i,t}(\zeta) = \mathbb{E}\left[\bar{g}_{i,t}(\zeta - D)\right]$, $i = 0, \cdots, l-2$, based on the expressions of $\bar{g}_{0,t}(\zeta), \cdots, \bar{g}_{l,t}(\zeta)$ shown in the proof of Proposition 5, we know that the expressions of $\bar{f}_{0,t}(\zeta), \cdots, \bar{f}_{l-2,t}(\zeta)$ are explicitly given (so are the functions $\bar{g}_{0,t}(\zeta), \cdots, \bar{g}_{l-1,t}(\zeta)$). Then, the rationing levels $r_{1,t}, \cdots, r_{l,t}$ can be directly obtained (see the definition of $r_{i,t}$ in the proof of Proposition 5). Note that demands are assumed to be i.i.d in different periods. The rationing levels are independent of t and hence can be denoted by $r_1, \cdots, r_l$.

The function $\bar{f}_{l-1,t}(\zeta)$ is expressed as

$$\bar{f}_{l-1,t}(\zeta) = c(\zeta \vee S_t) - c\zeta + \mathbb{E}\left[\bar{g}_{l-1,t}(\zeta - D)\right] + \mathbb{E}\left[\bar{g}_{l,t}(\zeta \vee S_t - D)\right].$$

When $\zeta \leq S_t$, based on the definition of S_t, we have

$$\bar{f}_{l-1,t}(\zeta) = U_t(\zeta) + \min_{v_l \geq \zeta} \mathbb{E}\left[cv_l + \bar{g}_{l,t}(v_l - D)\right],$$

where $U_t(\zeta) = -c\zeta + \mathbb{E}\left[\bar{g}_{l-1,t}(\zeta - D)\right]$, which is explicitly given and convex. Let $\hat{f}_{l-1,t}(\zeta) = \bar{f}_{l-1,t}(\zeta) - U_t(\zeta)$. Then,

$$\hat{f}_{l-1,t}(v_{l-1}) = \min_{v_l \geq v_{l-1}} \left(\hat{g}_t(v_l) + \beta \mathbb{E}\left[\hat{f}_{l-1,t+1}(v_l - D)\right]\right)$$

where

$$\begin{aligned} \hat{g}_t(v_l) =& cv_l + \mathbb{E}\left[(h + b_0)((v_l - D) \vee 0 - r_l \vee (v_l - D) \vee 0) + (s_{l+1} - b_0)(v_l - D - r_l \vee (v_l - D))\right] \\ &+ \mathbb{E}\left[\beta \sum_{k=1}^{l-1} \bar{f}_{k-1,t+1}(v_l - D) - \beta \sum_{k=1}^{l-1} \bar{f}_{k-1,t+1}(r_l \vee (v_l - D))\right] + \beta \mathbb{E}\left[U_{t+1}(v_l - D)\right]. \end{aligned}$$

Clearly, the expression of $\hat{g}_t(\zeta)$ is explicitly given and keeps the same under different t. Then,

$$\hat{S} = \min \arg\min_{\zeta} \hat{g}_t(\zeta)$$

is independent of t as demands are i.i.d in different periods. Note that in this case S_t can be equivalently obtained as $S_t = \min \arg\min_\zeta \left(\hat{g}_t(\zeta) + \beta \mathbb{E}\left[\hat{f}_{l-1,t+1}(\zeta - D)\right]\right)$. We assume that the terminal condition is $\hat{f}_{l-1,T+l+1}(\zeta) \equiv 0$ for any ζ.

Let $H_t(v_l) = \hat{g}_t(v_l) + \beta \mathbb{E}\left[\hat{f}_{l-1,t+1}(v_l - D)\right]$. Then, $\hat{f}_{l-1,t}(v_{l-1}) = \min_{v_l \geq v_{l-1}} H_t(v_l)$. We then show the optimality of the myopic policy by induction. In period $T + l$, $\hat{S} = S_{T+l}$ as $\hat{f}_{l-1,T+l+1}(\zeta) \equiv 0$ and $\hat{f}_{l-1,T+l}(v_{l-1}) = \hat{g}_{T+l}(\hat{S})$ when $v_{l-1} \leq \hat{S}$. Hence,

$$\hat{f}'_{l-1,T+l}(v_{l-1}) = \begin{cases} 0, & \text{if } v_{l-1} \leq \hat{S}, \\ H'_{T+l}(v_{l-1}) \geq 0, & \text{otherwise.} \end{cases}$$

Suppose that in period $t+1$ we have $\hat{f}'_{l-1,t+1}(v_{l-1}) = 0$ if $v_{l-1} \leq \hat{S}$ and $\hat{f}'_{l-1,t+1}(v_{l-1}) = H'_{t+1}(v_{l-1}) \geq 0$. It suffices to show that $\hat{f}_{l-1,t}(v_{l-1})$ has the same property. Similar to the argument in the proof of Proposition 3, we have that $S_t = \hat{S}$, $\hat{f}'_{l-1,t}(v_{l-1}) = 0$ if $v_{l-1} \leq \hat{S}$ and $\hat{f}'_{l-1,t}(v_{l-1}) = H'_t(v_{l-1}) \geq 0$. We thus have completed the inductive proof.

In summary, if the initial inventory level $v_{l-1,1} \leq \hat{S}$ in period 1, a myopic policy with the base stock level $\hat{S}$ defined above is optimal for the system with one demand class. ■

Appendix B

In this appendix, we briefly discuss how we can extend our results to other settings, in particular the serial systems. For simplicity, we can consider systems without expediting, but the results can be extended to systems with expediting by similar arguments.

Systems with Convex Backordering Costs

So far, we assume that the backordering costs are linear, which is a common assumption in the existing literature. In some cases, if the buyers have fill rate requirements, then the backordering costs might be convex. Specifically, we assume that the backordering cost for demand class j is $\hat{b}_j(w_j)$, where $\hat{b}_j$ is an increasing convex function of w_j. Assume that $\hat{b}_j$ is differentiable and the derivative of $\hat{b}_i$ is denoted by $\hat{b}'_j$. Moreover, we assume that $\lim_{w\to\infty}\hat{b}'_j(w) \geq \lim_{w\to 0}\hat{b}'_k(w)$ for any $k>j$, i.e. the marginal backordering cost for a high demand class should be no less than that of a low demand class. For this case, we still use the same state transformation as in (3) and (4). By a similar argument as in Lemma 2, we can show that it is optimal to fulfill demands with high marginal backordering costs first. Note that given $\boldsymbol{z}$, then the corresponding backordering cost is given by

$$\hat{b}(\boldsymbol{z}) = \sum_{j=1}^{n} \hat{b}_j(z_j - z_{j+1}) + \hat{b}_n(-z_n).$$

Note that since each $\hat{b}_j$ is a convex function, then it follows that the above function $\hat{p}$ is L$^\natural$-convex since

$$\hat{b}(\boldsymbol{z} - \eta\boldsymbol{e}) = \sum_{j=1}^{n} \hat{b}_j(z_j - z_{j+1}) + \hat{b}_n(-z_n + \eta).$$

is submodular in $(\boldsymbol{z},\eta)$.

Hence, by the same argument as in Lemma 5, we can show that the corresponding value functions under convex backordering costs are still L$^\natural$-convex. As a result, we can still show that Theorems 1 and 2 hold for the case with convex backordering costs as well.

Markov Modulated Demands

In this section, we show how we can extend our results to Markov modulated demands. There is a Markov chain ω_t, called the *world* as in Zipkin (2008). The distributions of demands of different classes in period t depend on the current world state ω_t. Let ω_+ be the world in the next state given current state ω. The dynamic recursion is given as follows:

$$f_t(\omega, \boldsymbol{w}, \boldsymbol{x}) = \min_{x_l \geq 0} \mathbb{E}[cx_l + g_t(\omega, \boldsymbol{w} + \boldsymbol{D}, \boldsymbol{x}, x_l)],$$

and

$$\begin{aligned} g_t(\omega, \bar{\boldsymbol{w}}, \bar{\boldsymbol{x}}) = & \min_{\bar{w}_i \geq a_i \geq 0, i=0,\cdots,n, \sum_{i=0}^n a_i \leq x_0} [\sum_{i=0}^{n} b_i(\bar{w}_i - a_i) + h(x_0 - \sum_{i=0}^{n} a_i) \\ & + f_{t+1}(w_+, \bar{w}_0 - a_0, \cdots, \bar{w}_n - a_n, x_0 + x_1 - \sum_{i=0}^{n} a_i, x_2, \cdots, x_l). \end{aligned} \tag{28}$$

We let $f_{T+l+1}(\omega, \boldsymbol{w}, \boldsymbol{x}) \equiv 0$. Notice that the only difference is now that the value function also depends on ω, but since ω is exogenous, we can still carry out the same structural analysis.

Systems with Fixed Ordering Intervals

In many cases, ordering may happen less frequently than inventory allocation since once there are demand realizations, the system manager must make the inventory allocation decisions. But most of our results still hold for this case. Since if there is no ordering opportunity for one period, we can set $y_t^* = v_{l-1,t}$, which would not affect our results (i.e., the L$^\natural$-convexity).

Stochastic Sequential Leadtimes

In this subsection we use the sequential leadtime model in Kaplan (1970) as in Zipkin (2008): at each period t, we observe the realization of a nonnegative random variable K_t and all orders that have been outstanding K_t or more periods arrive in period t. For simplicity, we assume that K_t are i.i.d. and $1 \le K_t \le l$. Suppose that we use $\boldsymbol{x}$ to describe the system state. Then x_0 is still the on-hand inventory level. But x_i denotes the order placed $l-i$ periods ago if that order has not arrived yet; certainly, if that order has arrived then $x_l = 0$. Again, as in Zipkin (2008), given state $x_{0,t}$ and the pipeline inventory $x_{i,t}, i = 1, \cdots, l-1$ and an ordering quantity q, the pipeline inventory state in the next period can be given by

$$\boldsymbol{x}_+ = \begin{cases} (x_0 + x_1, x_2, \cdots, x_{l-1}, q) & K_t = l, \\ (x_0 + x_1 + x_2, 0, x_3, \cdots, x_{l-1}, q) & K_t = l-1, \\ \cdots & \cdots \\ (x_0 + x_1 + \cdots + x_{l-1}, 0, \cdots, 0, q) & K_t = 2, \\ (x_0 + x_1 + \cdots + x_{l-1} + q, 0, \cdots, 0, 0) & K_t = 1. \end{cases}$$

Let $v_i = \sum_{k=0}^{i} x_k + z_0$ and $v_l = v_{l-1} + q$, and also define $D^n = \sum_{k=0}^{n} D_k$. The pipeline inventory positions in the next period are

$$\boldsymbol{v}_+ = \begin{cases} (v_1 - D^n, v_2 - D^n, \cdots, v_{l-1} - D^n, v_l - D^n) & K_t = l, \\ (v_2 - D^n, v_2 - D^n, v_3 - D^n, \cdots, v_{l-1} - D^n, v_l - D^n) & K_t = l-1, \\ \cdots & \cdots \\ (v_{l-1} - D^n, v_{l-1} - D^n, \cdots, v_{l-1} - D^n, v_l - D^n) & K_t = 2, \\ (v_l - D^n, v_l - D^n, \cdots, v_l - D^n, v_l - D^n) & K_t = 1. \end{cases}$$

Then the dynamic recursion for $\hat{f}_t$ is given as

$$\begin{aligned} \hat{f}_t(\boldsymbol{z}, \boldsymbol{v}) = \min_{v_l \ge v_{l-1}} \mathbb{E}\{& c(v_l - v_{l-1}) \\ &+ P(K_t = l)\hat{g}_t(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, v_1 - \sum_{k=0}^{n} D_k, \cdots, v_l - \sum_{k=0}^{n} D_k) \\ &+ P(K_t = l-1)\hat{g}_t(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, v_2 - \sum_{k=0}^{n} D_k, v_2 - \sum_{k=0}^{n} D_k, \\ & v_3 - \sum_{k=0}^{n} D_k, \cdots, v_{l-1} - \sum_{k=0}^{n} D_k, v_l - \sum_{k=0}^{n} D_k) \\ & \cdots \\ &+ P(K_t = 2)\hat{g}_t(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, v_{l-1} - \sum_{k=0}^{n} D_k, \cdots, v_{l-1} - \sum_{k=0}^{n} D_k, v_l - \sum_{k=0}^{n} D_k) \\ &+ P(K_t = 1)\hat{g}_t(z_n - D_n, \cdots, z_0 - \sum_{k=0}^{n} D_k, v_l - \sum_{k=0}^{n} D_k, \cdots, v_l - \sum_{k=0}^{n} D_k, v_l - \sum_{k=0}^{n} D_k)\}, \end{aligned} \tag{29}$$

and $\hat{g}_t$ is the same as before. Assuming $\hat{f}_{t+1}$ is L$^\natural$-convex, then every term here is also L$^\natural$-convex, and therefore so is the overall objective. Thus, the results of §4 remain to be valid.

Multiechelon Inventory Systems

We consider a serial system that consists of $N > 1$ stages (or echelons), indexed by $i = 1, \cdots, N$. All demands are met by the inventory of stage 1. Stage i replenishes the inventory from its immediate upstream stage $i+1$ for $i = 1, \cdots, N-1$, and stage N orders from an outside supplier with infinite supply. Without loss of generality, we assume that the leadtime from one stage to its immediate downstream stage is one period. Note that the leadtime is more than one, we can add a dummy stage for each leadtime period. We treat all outstanding orders as inventories at dummy stages. The function of each dummy stage is to ship all its inventory to the immediate downstream stage.

At the beginning of each period, we decide on the ordering quantity at each stage simultaneously before demand realization, while the inventory allocation decision is made afterwards. We assume that the ordering costs are linear and that there is no capacity limit at any stage. However, the inventory level in each upstream stage limits how much can be ordered from a downstream stage. Unfulfilled demands of all classes at stage 1 are fully backlogged. The objective is to minimize the total discounted cost by optimizing over inventory ordering decisions across stages and the inventory allocation decision at stage 1 over a finite horizon with T periods.

We use the following notation:

x_i = the inventory level at stage i after receiving deliveries at the beginning of each period;

q_i = the order quantity by stage i such that $q_i \in [0, x_{i+1}]$ for $i < N$, $q_N \geq 0$;

c_i = the unit ordering cost at stage i;

h_i= the unit holding cost at stage i.

We still use the same notion for the backorders of different demand classes and inventory allocation as in §4.

The state of the system is a $(n+N+1)$-dimension vector $(\boldsymbol{w}, \boldsymbol{x})$, where $\boldsymbol{w} = (w_0, \cdots, w_n)$ and $\boldsymbol{x} = (x_1, \cdots, x_N)$, which represents the system state at the beginning of each period after receiving deliveries. Let $\boldsymbol{q} = (q_1, \cdots, q_N)$ be the vector of ordering quantities from stages 1 to N and $\boldsymbol{a} = (a_0, \cdots, a_n)$ be the vector of allocation quantities to demand classes $0, \cdots, n$. After the ordering decision, the state before demand realization transits to

$$\boldsymbol{x}_+ = (x_1 + q_1, x_2 - q_2 + q_1, \cdots, x_i + q_i - q_{i-1}, \cdots, x_{N-1} + q_{N-1} - q_{N-2}, x_N + q_N - q_{N-1}).$$

Remark 3. Note that before demand realization, we do not receive the orders yet. However, in order to describe the ordering decision, we have to use the above dynamics. We can view these as the fictitious dynamics. The above dynamics would not change our cost since the same orders would be received at the beginning of the next period and the inventory cost in the current period is independent of the ordering quantities (See (19) and (20)).

Let $\boldsymbol{a}$ be the allocation policy such that $0 \leq a_j \leq w_j + D_j, j = 0, \cdots, n$ and $\sum_{j=0}^{n} a_j \leq x_1$, the backorder quantity in the next period is given by

$$\boldsymbol{w}_+ = (w_0 + D_0 - a_0, \cdots, w_n + D_n - a_n),$$

and the inventory state in the next period transits to

$$\boldsymbol{x}_{++} = (x_1 + q_1 - \sum_{j=0}^{n} a_j, x_2 + q_2 - q_1, \cdots, x_i + q_i - q_{i-1}, \cdots, x_N + q_N - q_{N-1}).$$

Let $\mathcal{A}(\boldsymbol{x}) = \{\boldsymbol{q} | 0 \le q_i \le x_{i+1}, i = 1, \cdots, N-1, q_N \ge 0\}$ be the feasible region. The dynamic recursion can be written as follows:

$$\tilde{f}_t(\boldsymbol{w}, \boldsymbol{x}) = \min_{\boldsymbol{q} \in \mathcal{A}(\boldsymbol{x})} \mathbb{E}[\sum_{i=1}^{N} c_i q_i + \tilde{g}_t(w_0 + D_0, w_1 + D_1, \cdots, w_n + D_n, \boldsymbol{x}_+)], \tag{30}$$

and

$$\begin{aligned} &\tilde{g}_t(w_0 + D_0, \cdots, w_n + D_n, \boldsymbol{x}_+) \\ = &\min_{w_j + D_j \ge a_j \ge 0, j=0,\cdots,n, \sum_{j=0}^n a_j \le x_1} \left[\sum_{j=0}^{n} b_j (w_j + D_j - a_j) + h_1 (x_1 - \sum_{j=0}^{n} a_j) + \sum_{i=2}^{N} h_i x_i + \beta \tilde{f}_{t+1}(\boldsymbol{w}_+, \boldsymbol{x}_{++}) \right]. \end{aligned} \tag{31}$$

We let $\tilde{f}_{T+l+1}(\boldsymbol{w}, \boldsymbol{x}) \equiv 0$.

Based on a similar argument as in Lemma 2, we are able to show that it is always optimal to fulfill a higher class demand first.

Next, we define $z_j = -\sum_{k=j}^n w_k, j = 0, \cdots, n$, $v_i = \sum_{k=1}^i x_k + z_0, i = 1, \cdots, N$. Let $y_i = v_i + q_i$ be the order-up-to decision at stage i, $i = 1, \cdots, N$. The the feasible region for the inventory ordering decision is given by

$$\mathcal{A}(\boldsymbol{v}) = \{\boldsymbol{y} | v_i \le y_i \le v_{i+1}, i = 1, \cdots, N-1, y_N \ge v_N\}.$$

Then the dynamics of $\boldsymbol{v}$ after the ordering decision but before demand realization is given by $(y_1, \cdots, y_N)$. Given the current state $(\boldsymbol{z}, \boldsymbol{v})$ and the allocation policy $\boldsymbol{a}$, in the next period the state transits to

$$z_{j+} = z_j - \sum_{k=j}^{n} (w_k + D_k - a_k).$$

Similar to the single stage model, the validity of the above transformation is due to the optimality of the priority fulfillment policy.

Under the state $(\boldsymbol{z}, \boldsymbol{v})$, we can rewrite the dynamic recursion:

$$\begin{aligned} &\bar{f}_t(z_n, \cdots, z_0, v_1, v_2, \cdots, v_N) \\ = &\min_{\boldsymbol{y} \in \mathcal{A}(\boldsymbol{v})} \mathbb{E}[\sum_{i=1}^{N} c_i (y_i - v_i) + \bar{g}_t(z_n + D_n, \cdots, z_0 + \sum_{k=0}^{n} D_k, y_1 - \sum_{k=0}^{n} D_k, \cdots, y_N - \sum_{k=0}^{n} D_k)], \end{aligned} \tag{32}$$

and $\bar{g}_t$ is defined as follows:

$$\bar{g}_t(\bar{z}_n, \cdots, \bar{z}_0, \bar{y}_1, \cdots, \bar{y}_N) = \min_{j=0,\cdots,n} F_{j,t}(\bar{z}_n, \cdots, \bar{z}_0, \bar{y}_1, \cdots, \bar{y}_n). \tag{33}$$

where $\bar{z}_j = z_j + \sum_{k=j}^n D_k$ for $j = 0, \cdots, n$, $\bar{y}_i = y_i - \sum_{k=0}^n D_k$ for $i = 1, \cdots, N$, and

$$\begin{aligned} &F_{j,t}(\bar{z}_n, \cdots, \bar{z}_0, \bar{y}_1, \cdots, \bar{y}_n) \\ = &\min_{\bar{z}_j \le u_j \le \min\{\bar{z}_{j+1}, \bar{y}_1\}} \left[\sum_{k=2}^{N} h_k (\bar{y}_k - \bar{y}_{k-1}) + h_1 (\bar{y}_1 - u_j) + \sum_{k=j+1}^{n} \hat{b}_k \bar{z}_k + b_j u_j \right. \end{aligned} \tag{34}$$

$$\bar{f}_{t+1}(\bar{z}_n,\cdots,\bar{z}_{j+1},\overbrace{u_j,\cdots,u_j}^{j+1},\bar{y}_1,\cdots,\bar{y}_N)\Bigg].$$

Similarly, we define $\bar{f}_{T+l+1}(\boldsymbol{z},\boldsymbol{y})\equiv 0$.

The formulations in (32) and (33) are similar to the formulations in (7) and (8) in the main body, respectively. In order to describe the optimal inventory allocation policy, given the realized demand $(D_0,\cdots,D_n)$, we define $(\bar{\boldsymbol{z}},\bar{\boldsymbol{y}})=(\bar{z}_n,\cdots,\bar{z}_0,\bar{y}_1,\cdots,\bar{y}_N)$ and $(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=(w_0+D_0,\cdots,w_n+D_n,x_1+q_1,\cdots,x_i+q_i-q_{i-1},\cdots,x_N+q_N-q_{N-1})$.

Let $y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$, $i=1,\cdots,N$ and $u^*_{j,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{y}})$, $j=1,\cdots,N$ be the least optimal solutions of the dynamic programs in (32) and (33), respectively, and $q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$ be the optimal ordering quantity that corresponds to $y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$, i.e., $q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})=y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})-v_{l-1}$. Also, we let $q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})=q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$, and $a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=a^*_{j,t}(\bar{\boldsymbol{z}},\bar{\boldsymbol{y}})$.

Next, we show various monotone properties of the optimal policy based on L$^\natural$-convexity.

Theorem 4. *$\bar{f}_t,\bar{g}_t$ are $L^\natural$-convex for all t. Moreover, for $\delta>0$, the following results hold:*

(1) For $i=1,\cdots,N$, $y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $(\boldsymbol{z},\boldsymbol{v})$ and

$$\begin{aligned}
-\delta \le & q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_i)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{i-1})-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})\le\cdots\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})\\
= & q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})-q^*_{i,t}(\boldsymbol{w}+\delta\boldsymbol{e}_1,\boldsymbol{x})\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})-q^*_{i,t}(\boldsymbol{w}+\delta\boldsymbol{e}_2,\boldsymbol{x})\le\cdots\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})-q^*_{i,t}(\boldsymbol{w}+\delta\boldsymbol{e}_{n+1},\boldsymbol{x})\le 0,\\
0\le & q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_N)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{N-1})-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x})\le\cdots\le q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{i+1})-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}).
\end{aligned}$$

(2) for $j=0,\cdots,n$

$$\begin{aligned}
&-\delta\le a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_j,\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})=\cdots=a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_1,\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\le 0\\
&\delta\ge a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{j+1},\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge\cdots\ge a^*_{j,t}(\bar{\boldsymbol{w}}+\delta\boldsymbol{e}_{n+1},\bar{\boldsymbol{x}})-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0,\\
&\delta\ge a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_1)-a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge\cdots\ge a^*_{j,t}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}}+\delta\boldsymbol{e}_N)-a^*_{jt}(\bar{\boldsymbol{w}},\bar{\boldsymbol{x}})\ge 0.
\end{aligned}$$

The theorem shows that the optimal ordering quantity at any stage j is non-increasing in the net inventory level at any downstream stage $k\le j$ and increasing in the net inventory level at any upstream stage $j<k$. All of the sensitivities are bounded by 1. Moreover, the sensitivity of the optimal ordering quantity at any stage j to the net inventory level at a downstream stage is greater than that at a further downstream stage. The rest of the properties are similar to those in Theorem 2.

A Sketch of Proof for Theorem 4

Based on a similar argument as in Theorem 1, we can show that $\bar{f}_t(\boldsymbol{z},\boldsymbol{v})$ and $\bar{g}_t(\boldsymbol{z},\boldsymbol{v})$ are L$^\natural$-convex since $\mathcal{A}(\boldsymbol{v})$ is a sublattice.

From Lemma 5 (c), we know that $y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $(\boldsymbol{z},\boldsymbol{v})$ and satisfies

$$y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})\le y^*_{i,t}(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e})\le y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})+\delta.$$

Since $y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})=q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})+v_i$, the above analysis implies that $q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})$ is nondecreasing in $v_k,k\neq i$ and $\boldsymbol{z}$. Moreover,

$$q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i)=y^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i)-\delta-v_i\ge y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})-\delta-v_i=q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})-\delta.$$

Hence, we have $q^*_{i,t}(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \geq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}) \ \geq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i) \geq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v})-\delta$. Finally, $y^*_{i,t}(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) \leq y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})+\delta$ implies that

$$q^*_{i,t}(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{v}+\delta\boldsymbol{e}) = y^*_{i,t}(\boldsymbol{z}+\delta\boldsymbol{e},\boldsymbol{z}+\delta\boldsymbol{e})-\delta-v_i \leq y^*_{i,t}(\boldsymbol{z},\boldsymbol{v})-v_i = q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}).$$

In summary, we have

$$\begin{aligned} 0 \leq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_k)-q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}) \leq \delta, k\neq j,\\ -\delta \leq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e}_i)-q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}) \leq 0,\\ -\delta \leq q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}+\delta\boldsymbol{e})-q^*_{i,t}(\boldsymbol{z},\boldsymbol{v}) \leq 0. \end{aligned}$$

If we translate these inequalities into the original state, we have

$$\begin{aligned} -\delta \leq & q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_i)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}) \leq q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{i-1})-q^*_{j,t}(\boldsymbol{w},\boldsymbol{x}) \leq \cdots \leq q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_1)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}),\\ 0 \leq & q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_N)-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}) \leq q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{N-1})-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}) \leq \cdots \leq q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}+\delta\boldsymbol{e}_{i+1})-q^*_{i,t}(\boldsymbol{w},\boldsymbol{x}). \end{aligned}$$

The rest of the proof can be accomplished by a similar argument as in Theorems 1 and 2. ■

References

Boyd, S., L. Vandenberghe. 2004. *Convex optimization*. Cambridge University Press.

Heyman, D., M. Sobel. 1984. *Stochastic Models in Operations Research*, Vol. II. McGraw Hill, New York.

Kaplan, R. 1970. A dynamic inventory model with stochastic lead times. *Management Science* 16 491-507.

Topkis, D. 1998. *Supermodularity and Complementarity*, Princeton University Press.

Zipkin, P. 2008. On the structure of lost-sales inventory model. *Operations Research* 56(4) 937-944.