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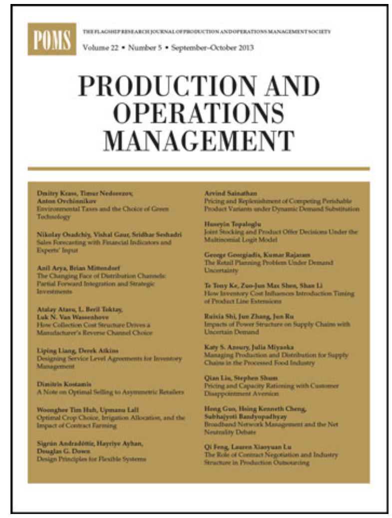

Title: Coordinating Inventory and Pricing Decisions with General Price-Dependent Demands

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Coordinating Inventory and Pricing Decisions with General Price-Dependent Demands*

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We consider a periodic-review, joint inventory and pricing control problem for a firm that faces general random price-dependent demands. Any unsatisfied demand can be either backordered or lost immediately. The objective is to maximize the expected profit over a finite selling horizon by coordinating the inventory and pricing decisions in each period. For both the backorder model and the lost-sales model, we derive some quite general sufficient conditions to ensure the optimality of a base-stock list price (BSLP) policy based on the strict monotonicity of demand functions in the realizations of random noises. We are among the first to utilize the strict monotonicity of demand functions in the realizations of random noises for deriving the sufficient conditions. We derive the sufficient conditions in both the backorder model and the lost-sales model by utilizing the new concept of upper-set and lower-set decreasing properties (USDP/LSDP), which is a generalized version of the first-order stochastic dominance. This study reveals that the optimality of a BSLP policy is robust to more general business environments than what we previously thought. Finally, we also apply the USDP/LSDP in other inventory management problems.

Key words: inventory/pricing; lost-sales; general demand model; dynamic programming
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1. Introduction

Pricing strategies become increasingly important in retail and manufacturing sectors due to the development of information technology. Firms can easily change their prices based on demand seasonality, inventory levels, and production schedules, etc. In practice, the inventory-based dynamic pricing is adopted by firms like Dell, Amazon, FairMarket, Land's End, and J.C. Penney (Elmaghraby and Keskinocak 2003, Chan et al. 2004). Through coordinating inventory and pricing strategies, significant benefits can be reaped for these firms (Chen and Simchi-Levi 2012).

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To be implementable in practice, a joint inventory and pricing strategy is required to have a simple structure. An appealing candidate is the so-called base-stock list price (BSLP) policy. Under this policy, there is a base-stock level. Whenever the inventory level is below the base-stock level, the firm orders up to that level and charges a list price. Otherwise, no order is placed and the price is decreasing in the inventory level. However, this simple policy may not always be optimal. To ensure the effectiveness of a BSLP policy, it is thus critical to identify conditions and business environments under which a BSLP policy is optimal. The optimality conditions provide valuable benchmarks for the implementation of a BSLP policy in practice.

The joint inventory and pricing control has received considerable attention over the past ten years. Validating the optimality of a BSLP policy is one of the focuses in the existing literature. However, the existing literature suffers the following two limitations: (1) for the backorder setting, it requires unnecessarily restrictive functional forms of demand; (2) the lost-sales setting is suitable for describing the business model of retail, but its corresponding results are very limited and hence this setting remains largely unknown. Specifically, for the backorder setting, there are mainly three classes of demand functions considered in the existing literature: additive functions, multiplicative functions and generalized additive functions. Beyond these three classes, our understanding is very limited for the optimality of a BSLP policy. Under the lost-sales setting, in the existing literature, we do not know when a BSLP policy is optimal in a finite selling horizon even under the additive demand functions due to its technical challenges, not to mention the general demand functions. This raises doubts on the applicability of a BSLP policy in practice.

The two limitations mentioned above mainly result from the technical challenges caused by the general demand functions and the revenue function in the lost-sales setting. More specifically, with the general demand functions, it is challenging to show the joint concavity of the single-period expected profit function under the backorder setting because the composition of two concave functions in general is not concave. For the lost-sales setting, the revenue function in each period is censored by the inventory level, and as a result it may not be concave. To overcome these technical challenges, we utilize the monotonicity of demand functions in the realizations of random noises.

The focus of our paper is to identify the conditions for the optimality of a BSLP policy under a finite selling horizon with general demand functions. We do not impose any particular form on how demand depends on price and random noise. Consistent with the existing inventory-pricing literature, we consider two classical inventory models: the backorder inventory model and the lost-sales inventory model with a single stage and zero leadtime. By adopting the monotonicity of demand functions introduced above, we are able to identify sufficient conditions to ensure the optimality of a BSLP policy under general demand functions.

Specifically, we show for the backorder setting that a BSLP policy is optimal in each period if demand functions are decreasing in price and strictly decreasing (increasing) in the realizations of random noises, and their sensitivities in price have the upper-set (lower-set) decreasing property (USDP/LSDP). The new concept of USDP/LSDP is in fact a generalized version of the first-order stochastic dominance. The demand sensitivity in price has the LSDP (USDP) if it is stochastically increasing (decreasing) in price. If a demand function is concave in price, then its sensitivity in price must have the USDP and LSDP. However, the concavity or convexity in price is not necessary to ensure the USDP/LSDP.

For the lost-sales setting, we show that a BSLP policy is optimal in each period if (i) demand functions are decreasing in price and strictly decreasing (increasing) in the realizations of random noises, and their sensitivities in price have the USDP (LSDP), and (ii) the single-period expected profit function is submodular in price and inventory level. Under the single-period setting, Kocabiyikoğlu and Popescu (2011) show that the sufficient conditions for the optimality of a BSLP policy are that the demand function is decreasing in price and increasing in the realizations of random noises, the riskless unconstrained revenue function is concave in price for any realizations of random noises, and the single-period expected profit function is submodular in price and inventory level. We show that, unlike the single-period setting, we require the USDP/LSDP of the sensitivity of demand functions in price in the multi-period setting, which also guarantees the concavity of the expected riskless unconstrained revenue function in price. We do not require that the revenue function is concave in price for any realizations of random noises. In summary, in spite of the technical challenges, we find that a BSLP policy is optimal for both the backorder inventory model and the lost-sales inventory model under quite general conditions.

The proposed concept of USDP/LSDP can be applied to other operations management problems. We show for example in both the backorder model and the lost-sales model, with inventory-dependent demands, the USDP/LSDP can be used to guarantee that an inventory-dependent base-stock policy is optimal; with quality-dependent demands, the USDP/LSDP can lead to the optimality of the base-stock list quality level policy.

In summary, our contributions include:

(1) We consider general price-dependent demands in a multi-period setting and show that a BSLP policy is optimal under more general sufficient conditions compared with the conditions proposed in the existing literature for both the backorder model and the lost-sales model. We are

among the first to derive results by utilizing the strict monotonicity of demand functions in the realizations of random noises.

(2) To show the joint concavity of our objective functions in both the backorder model and the lost-sales model, we propose the concept of USDP/LSDP, show the preservation of its monotonicity under the expectation with a class of monotone functions, and exploit the monotonicity of demand functions in the realizations of random noises. In addition, based on the USDP/LSDP, we show the concavity of a function that is a composite of a concave function with another function having the USDP/USLP.

2. Related Literature

There is an extensive literature on joint pricing and inventory control strategies starting with Whitin (1955). Whitin (1955) analyzes an EOQ model with deterministic (price-dependent) demand. On models with stochastic demand, there are two streams of literature: one focuses on single-period problems and the other focuses on multi-period problems. See Petruzzi and Dada (1999), Elmaghraby and Keskinocak (2003), Yano and Gilbert (2004), and Chen and Simchi-Levi (2012) for comprehensive reviews. The early single-period coordinating inventory and pricing control models include Mills (1959, 1962), Karlin and Carr (1962), Zabel (1970), Young (1978), Cheng (1984), Lau and Lau (1988), and Polatoglu (1991). Recent studies on single-period models include Yao et al. (2006), Kocabiyikoğlu and Popescu (2011), and Lu and Simchi-Levi (2013). In particular, Kocabiyikoğlu and Popescu (2011) consider the lost-sales model in a single-period setting and provide sufficient conditions to ensure the concavity and submodularity of the objective functions.

One stream of literature focuses on multi-period models under the backorder setting. Zabel (1972) analyzes both the additive and multiplicative demand functions where the random noise follows a uniform or exponential distribution that is independent of price. Thowsen (1975) extends Zabel's (1972) results for the additive demand function by assuming the random noise follows a Pólya frequency function of order 2 (PF2) distribution and finds that a BSLP policy is optimal. Federgruen and Heching (1999) establish the optimality of a BSLP policy with a general demand function for a backorder system without capacity constraints. Their sufficient conditions are as follows: (1) the demand is decreasing and concave in the list price; (2) the single-period expected inventory cost is jointly convex in the order-up-to level and price. However, as indicated by the authors themselves and Feng et al. (2013), the joint convexity of the single-period expected inventory cost function can be guaranteed if the demand function is linear in the price, which is quite restrictive, but easily fails when the demand is a nonlinear function of the price.

Due to the technical challenges caused by the general demand function, the subsequent literature mainly considers the following classes of demand functions: additive demand function, multiplicative demand function, combined additive and multiplicative demand function (Chen and Simchi-Levi 2004a), and generalized additive demand function (Feng et al. 2013). In fact, both Chen and Simchi-Levi (2004a) and Feng et al. (2013) consider a mixture of additive and multiplicative demand functions.

Chen and Simchi-Levi (2004a, b) generalize Federgruen and Heching's (1999) model by incorporating a positive and fixed ordering cost, and relaxing the concave demand assumption. Based on the combined additive and multiplicative demand function, they prove that an (s, S, p) policy is optimal for the additive demand function and an (s, S, A, p) policy is optimal for the combined demand function. Chen et al. (2010), based on the additive demand function, consider the concave ordering cost and show that a generalized (s, S, p) policy is optimal if the random noise in a demand function follows a positive Pólya or uniform distribution. Chen and Zhang (2014) analyze a nonstationary inventory system with the additive demand function and show that a time-dependent (s, S, p) policy is optimal.

Feng et al. (2013) consider the generalized additive demand function, which involves the relationship between scale and location parameters. They show that a BSLP policy is optimal under a new set of optimality conditions that depend on the location and scale parameters of the demand. Bernstein et al. (2016) consider both additive and multiplicative demand functions for systems with a positive lead time. They propose a simple heuristic consisting of a myopic pricing policy and a base-stock policy for replenishment. Lu et al. (2016) assume a combined additive and multiplicative demand function and consider that the demand information is incomplete in reality. By introducing a new concept named K-approximate convexity, they obtain a base-stock list-price policy with a good performance. An intriguing research question is: Can we identify general sufficient conditions beyond the aforementioned classes of demand functions to ensure the optimality of a BSLP policy under the backordering setting?

There are only a few studies that consider the multi-period lost-sales setting. This is due to the technical difficulty that the demand is censored by the inventory level. For a system with fixed ordering cost and general demand functions, Polatoglu and Sahin (2000) provide some sufficient conditions for an (s, S, p) policy to be optimal. But, as indicated by Chen and Simchi-Levi (2004a), it is not clear if their conditions can be satisfied by any demand function. For example, a linear demand function does not necessarily match those requirements. When the demand is in the additive form, Chen et al. (2006) establish the optimality of an (s, S, p) policy by imposing some restrictions on the distribution of demand uncertainty as well as some restrictions on the inequality involving expected demand and price. When the demand is in the multiplicative form, Song et al. (2009) demonstrate the optimality of an (s, S, A, p) policy for a finite horizon problem. Furthermore, the optimal policy can be simplified to a base-stock policy when the fixed ordering cost is zero. For both the lost-sales model and the backorder model, Huh and Janakiraman (2008) use an alternative approach to investigate the sufficient conditions for the optimality of an (s, S, p) policy based on the combined additive and multiplicative demand function.

Different from the existing literature, to show the optimality of a BSLP policy in a multi-period setting, we utilize the monotonicity of general demand functions in the realizations of random noises. We are able to derive a set of general sufficient conditions for the lost-sales setting. A more relax set of sufficient conditions are obtained for the backorder setting.

The remainder of the paper is organized as follows. We describe our model settings in Section 3 and present some preliminary results in Section 4. Section 5 provides formulations and analytical results for the backorder model and the lost-sales model, respectively. Section 6 discusses some applications of our results and proposed concepts. Finally, we provide concluding remarks in Section 7. All proofs and some of the intermediate results are relegated to Appendix A.

3. Model Settings

We consider a single-product single-stage periodic-review inventory problem with general price-dependent demands. The finite selling horizon consists of T periods. The demand in each period only depends on the prevailing list (selling) price, i.e., customers are myopic. Following Federgruen and Heching (1999), we assume the demand in period t is $d_t(p_t, \epsilon_t)$, where p_t denotes the list price and ϵ_t is a continuous random noise of the demand with a probability density function $w_t(\zeta)$ and accordingly a cumulative distribution function $W_t(\zeta)$ for $\zeta \in (-\infty, +\infty)$ in each period t, $t = 1, \cdots, T$. In addition, for each t, $p_t \in [\underline{p}, \overline{p}]$ and ϵ_t's are stochastically independent across different periods. For simplicity, we assume that the support of ϵ_t is $[\underline{\epsilon}, \overline{\epsilon}]$ (see Appendix B for the case with unbounded support). Consistent with the inventory-pricing literature, we assume zero leadtime, i.e., once an order is placed, it arrives in the same period. For the backorder/lost-sales setting, we denote by x_t the net inventory/on-hand inventory level at the beginning of period t, q_t the ordering quantity, and $y_t = x_t + q_t$ the order-up-to level after the ordering decision in period t. There is no limit on how much we can order, and there are a unit ordering cost c and a unit salvage cost α at the end of the selling horizon. Following Huh and Janakiraman (2008), we normalize the unit ordering cost as 0, i.e., $c = 0$, since a system with a positive unit ordering cost can be equivalently transformed into the one in which the ordering cost is zero and the other cost parameters are suitably modified.

The sequence of events in each period is as follows: (1) at the beginning of each period, an ordering decision and a pricing decision are made simultaneously; (2) the order in this period arrives; (3) demand is realized after customers observe the list price; (4) finally, the revenue/cost is charged at the end of the period. For unfilled demand, we consider either the backorder setting or the lost-sales setting.

To analyze the inventory-pricing problems, we impose the following two assumptions on the demand functions.

Assumption 1. *For all t, $t=1,\cdots,T$, (1) the demand function $d_t(p_t,\zeta)$ is decreasing in p_t and strictly monotone in ζ, where ζ denotes the realization of ϵ_t; (2) $\mathbb{E}[d_t(p_t,\epsilon_t)]<+\infty$ for $\underline{p}\leq p_t\leq\overline{p}$ is strictly decreasing in p_t.*

The fact that $d_t(p_t,\zeta)$ is decreasing in p_t is a regular assumption on demand functions, i.e., as price increases, demand should decrease.

Unlike the existing literature, the monotonicity of $d_t(p_t,\zeta)$ in ζ plays a critical role in our analysis. In fact, under the multiplicative or additive demand models, we already assume some monotone properties for the random noises. However, the monotonicity is not utilized in deriving the structural properties of the optimal profit functions in the existing literature.

Since $d_t(p_t,\zeta)$ is strictly monotone in ζ, there exists some $\bar{\zeta}_t$ such that $y_t-d_t(p_t,\bar{\zeta}_t)=0$ for given (p_t,y_t) in period t, i.e., the realized noise under which the demand is equal to the order-up-to level in period t. Let $d_t^{-1}(p_t,y_t)$ be the inverse function of $d_t(p_t,\epsilon_t)$ with respect to (w.r.t) ϵ_t, i.e., $\bar{\zeta}_t=d_t^{-1}(p_t,y_t)$. Then, $\bar{\zeta}_t$ is a function of the decision variables (p_t,y_t).

Assumption 2. *$d_t(p_t,\epsilon_t)$ is thrice continuously differentiable in (p_t,ϵ_t). The inverse function $d_t^{-1}(p_t,y_t)$ and the probability density function $w_t(\zeta)$ are continuously differentiable in (p_t,y_t) and ζ, respectively.*

We are not the first to assume the thrice continuous differentiability of the demand function. Chen et al. (2006) and Huh and Janakiraman (2008) make similar assumptions to derive sufficient conditions for the optimality of an (s,S) policy under the lost-sales model. Assumption 2 is needed to ensure the twice continuous differentiability of the expected discounted profit functions in the backorder and the lost-sales models. We denote by $d_{t,p}(p_t,\zeta)=\frac{\partial d_t(p_t,\zeta)}{\partial p_t}$ and $d_{t,\zeta}(p_t,\zeta)=\frac{\partial d_t(p_t,\zeta)}{\partial\zeta}$ the first order derivatives w.r.t p_t and ζ, respectively.

Based on the model setting discussed above, we then introduce a new concept and present some preliminary results that shall be used to derive our main results.

4. Preliminary Results

To empower our analysis, we introduce the notion of upper-set (lower-set) decreasing property as follows.

Definition 1. A function $f(x,\epsilon)$, where ϵ is a random variable with the density function $\omega(\cdot)$ and $x \in A \subseteq \mathbb{R}$, has the upper-set decreasing property (USDP) if $\int_{\mu}^{\overline{\epsilon}} f(x,\zeta)\omega(\zeta)d\zeta$ is decreasing in x, $x \in A$, for any $\mu \in [\underline{\epsilon}, \overline{\epsilon}]$. Similarly, a function $f(x,\epsilon)$ is said to have the lower-set decreasing property (LSDP) if $\int_{\underline{\epsilon}}^{\mu} f(x,\zeta)\omega(\zeta)d\zeta$ is decreasing in x for any $\mu \in [\underline{\epsilon}, \overline{\epsilon}]$.
The USDP/LSDP is similar to the first-order stochastic dominance for random variables. However, $f(x,\epsilon)$ may not be a density function as it might be negative.

Note that if $f(x,\epsilon)$ is decreasing in x for any realization of ϵ, then it must have the USDP/LSDP. But a function $f(x,\epsilon)$ having the USDP/LSDP is not necessarily decreasing in x for any ϵ. To illustrate this point, we provide an example of USDP as follows:

Example 1. Let $f(x,\epsilon) = x^{\epsilon}\ln(x)$, where ϵ is uniformly distributed over $[1, M]$ and $0 \leq x \leq e^{-\ln(M)/(M-1)}$. Then, $f(x,\epsilon)$ is not always decreasing in x for any realization of ϵ. Instead, $f(x,\epsilon)$ is decreasing in x when $x \to 0$ for any ϵ but may be increasing in x when $x \to e^{-\ln(M)/(M-1)}$ and the realization of ϵ is small. Note that $\int_{\mu}^{M} x^{\zeta}\ln(x)\frac{1}{M-1}d\zeta = \frac{1}{M-1}(x^M - x^{\mu})$. Its derivative $\frac{1}{M-1}(Mx^{M-1} - \mu x^{\mu-1}) \leq 0$ for $0 \leq x \leq e^{-\ln(M)/(M-1)}$ because $\ln(x) \leq \frac{-\ln(M)}{M-1}$ based on the bound of x, $\frac{-\ln(M)}{M-1} \leq \frac{\ln(\mu)-\ln(M)}{M-\mu}$ as $\frac{\ln(\mu)-\ln(M)}{M-\mu}$ is increasing in μ (this monotone property holds due to $\left(\frac{\ln(\mu)-\ln(M)}{M-\mu}\right)' = \frac{M/\mu-1-\ln(M/\mu)}{(M-\mu)^2}$ and $\ln(M/\mu) < M/\mu - 1$).

A function with the USDP/LSDP leads to the following results.

Lemma 1. *For a continuous random variable ϵ with a density function $\omega(\cdot)$, $f(x,\epsilon)$ has the USDP if and only if*

$$\int_{\mu}^{\overline{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta \tag{1}$$

for any $\mu \in [\underline{\epsilon}, \overline{\epsilon}]$ is increasing in x for any non-positive and decreasing function $g(\cdot)$. Similarly, $f(x,\epsilon)$ has the LSDP if and only if $\int_{\underline{\epsilon}}^{\mu} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ for any $\mu \in [\underline{\epsilon}, \overline{\epsilon}]$ is increasing in x for any non-positive and increasing function $g(\cdot)$.

Lemma 1 shows that the integration in (1) can preserve the monotonicity for any non-positive and increasing/decreasing function $g(\cdot)$ under certain conditions. This result is new in the existing literature.

Remark 1. Note that $f(x,\epsilon)$ is stochastically increasing (decreasing) in x if and only if

$$\int_{\underline{\epsilon}}^{\overline{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$$

is increasing (decreasing) in x for *any increasing* function $g(\cdot)$ (see Shaked and Shanthikumar, 1994). In this sense, if $f(x,\epsilon)$ is stochastically increasing (decreasing) in x, then it must have the LSDP (USDP). Hence, the USDP/LSDP is a generalization of the first-order stochastic dominance.

Based on Lemma 1, we then show that the USDP/LSDP can also be used to ensure the joint concavity of the expected value of a composite function.

PROPOSITION 1. *Let* $F(x,z) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} f(x - \psi(z,\zeta))\omega(\zeta)d\zeta$, *where* $f(\cdot)$ *is twice continuously differentiable,* $\psi(z,\zeta)$ *is thrice continuously differentiable, and* ζ *is the realization of the random noise* ϵ *in the support* $[\underline{\epsilon},\overline{\epsilon}]$. *The first order derivative of* $\psi(z,\zeta)$ *w.r.t* z *is denoted by* $\psi_z(z,\zeta)$. *Then,* $F(x,z)$ *is jointly concave in* (x,z) *under the following conditions:*

(i) $f(\cdot)$ *is a decreasing and concave function;*

(ii) $\psi(z,\zeta)$ *is strictly decreasing (increasing) in* ζ *and* $\psi_z(z,\zeta)$ *has the USDP (LSDP).*

Proposition 1 is critical for the analysis of the backorder and the lost-sales models. It is worth noting that, in this proposition, we do not require $\psi(z,\zeta)$ being concave in z for any ζ to ensure the joint concavity of $F(x,z)$. Proposition 1 is also useful for analyzing operations management models where demand could depend on some other factors, e.g., inventory, quality, etc.

Based on Proposition 1, if a single-variable function $f(\cdot)$ is concave and increasing, then $\mathbb{E}_\zeta[f(g(x,\zeta))] = \int_{\underline{\epsilon}}^{\overline{\epsilon}} f(g(x,\zeta))w(\zeta)d\zeta$ is concave in x when $g(x,\zeta)$ is strictly decreasing (increasing) in ζ and $g_x(x,\zeta)$ has the USDP (LSDP). The strict monotonicity of $g(x,\zeta)$ in ζ is required to ensure the decreasing property of $f'(g(x,\zeta))$ in ζ so that we can adopt Lemma 1 to derive the result. Note that, for a concave function $f(\cdot)$, a sufficient condition to ensure the concavity of the composite function $\mathbb{E}_\zeta[f(g(x,\zeta))]$ in x is that $f(\cdot)$ is strictly decreasing (increasing) and $g(x,\zeta)$ is convex (concave) in x for any ζ. It seems that, to preserve the concavity of $\mathbb{E}_\zeta[f(g(x,\zeta))]$, we need the monotonicity of $f(\cdot)$. However, with the USDP/LSDP condition, we no longer require $g(x,\zeta)$ to be convex or concave in x for any ζ. Instead, we replace this restrictive condition by exploiting the monotonicity of $g(x,\zeta)$ in ζ and the USDP/LSDP of $g_x(x,\zeta)$ to ensure the concavity of $\mathbb{E}_\zeta[f(g(x,\zeta))]$. For a multi-dimensional vector $\mathbf{x}$ and the function $g(\mathbf{x},\zeta)$, we may follow the analysis as in Proposition 1 and derive sufficient conditions for the concavity of the composite function $\mathbb{E}_\zeta[f(g(\mathbf{x},\zeta))]$ based on the USDP/LSDP. However, the conditions shall depend on the specific form of $g(\mathbf{x},\zeta)$ and hence it is out of our scope to discuss them in this study.

5. The Optimality of a BSLP Policy

With the preliminary results above, we now are ready to analyze the backorder model and the lost-sales model sequentially. For each of the models, we present individually the dynamic programming formulations and the analytical results. For the ease of exposition, we make the following assumption in subsequent analysis:

ASSUMPTION 3. *The demand function $d_t(p_t, \epsilon_t)$ is strictly decreasing in the realizations of ϵ_t.*

Note that we do not impose any particular class of distributions for ϵ_t. If $d_t(p_t, \zeta)$ is strictly increasing in ζ, then $d_t(p_t, -\psi)$ where $\psi = -\zeta$ must be strictly decreasing in ψ. Hence, Assumption 3 is made without loss of generality for the monotonicity of demand functions as in Assumption 1.

5.1. The Backorder Model

In the backorder model, unfulfilled demand in each period is backordered and leftover inventory is carried over to the next period. Given the initial state x_t, the order quantity q_t, and the list price p_t in period t, the state at the beginning of period $t+1$ transits to

$$x_{t+1} = x_t + q_t - d_t(p_t, \epsilon_t) = y_t - d_t(p_t, \epsilon_t),$$

where $y_t = x_t + q_t$ is the order-up-to level as defined before.

There are a unit backordering cost b for backordered demand and a unit holding cost h for leftover inventory. Then, the single-period inventory cost incurred in period t is denoted by

$$\mathcal{C}(y_t - d_t(p_t, \zeta)) = h[y_t - d_t(p_t, \zeta)]^+ + b[d_t(p_t, \zeta) - y_t]^+,$$

which is convex, where $[z]^+ = \max\{0, z\}$. The revenue/cost in future periods is discounted with a discount factor $\beta \in [0, 1]$.

Let $\widehat{V}_t(\cdot)$ be the expected optimal discounted profit function from period t and onward. The dynamic recursion for this problem is:

$$\widehat{V}_t(x_t) = \max_{\underline{p} \le p_t \le \overline{p},\, y_t \ge x_t} \widehat{U}_t(p_t, y_t), \tag{2}$$

where the objective function $\widehat{U}_t(p_t, y_t)$ is

$$\widehat{U}_t(p_t, y_t) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \Big[p_t d_t(p_t, \zeta) - \mathcal{C}(y_t - d_t(p_t, \zeta)) + \beta \widehat{V}_{t+1}(y_t - d_t(p_t, \zeta)) \Big] w_t(\zeta) d\zeta \tag{3}$$

for $t = 1, \cdots, T$. Finally, $\widehat{V}_{T+1}(x_{T+1}) \equiv \alpha x_{T+1}$ for all x_{T+1}, where α is the unit salvage value.

We then shall provide sufficient conditions to ensure that the optimal policy in the backorder model can be described by a BSLP policy, i.e., the ordering policy is a base-stock policy and the price is decreasing in the order-up-to level. Essentially, we seek to establish the joint concavity and submodularity in the order-up-to level and the list price for the objective function $\widehat{U}_t(p_t, y_t)$ in each period. Under general demand functions, it is difficult to ensure the concavity of $\widehat{V}_t(\cdot)$. We need to identify the classes of demand functions and cost parameters that ensure the optimality of a BSLP policy. Our method involves an equivalent transformation made to the dynamic recursion of this problem, and validating the submodularity and the joint concavity (more specifically the negative semidefinite of the corresponding Hessian matrix) of the transformed objective function in our multi-period setting based on an inductive proof.

To facilitate our analysis for the backorder model, we first make an equivalent transformation to the dynamic recursion of this problem as follows:

$$\widehat{V}_t(x_t) + Mx_t = \max_{\underline{p} \leq p_t \leq \overline{p},\, y_t \geq x_t} \int_{\underline{\epsilon}}^{\overline{\epsilon}} [p_t d_t(p_t,\zeta) + Mx_t - \mathcal{C}(y_t - d_t(p_t,\zeta)) - \beta M(y_t - d_t(p_t,\zeta)) + \beta \widehat{V}_{t+1}(y_t - d_t(p_t,\zeta)) + \beta M(y_t - d_t(p_t,\zeta))] w_t(\zeta) d\zeta$$

for any constant M. This cost accounting scheme only shifts costs across time periods and hence does not affect the total profit over the entire selling horizon. Let $V_t(x_t) = \widehat{V}_t(x_t) + Mx_t$. Then, the above dynamic recursion becomes

$$V_t(x_t) = \max_{\underline{p} \leq p_t \leq \overline{p},\, y_t \geq x_t} [U_t(p_t, y_t) + Mx_t] \tag{4}$$

where

$$U_t(p_t,y_t) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} [(p_t + \beta M) d_t(p_t,\zeta) - \beta M y_t - \mathcal{C}(y_t - d_t(p_t,\zeta)) + \beta V_{t+1}(y_t - d_t(p_t,\zeta))] w_t(\zeta) d\zeta, \tag{5}$$

and $V_{T+1}(x_{T+1}) = (M+\alpha)x_{T+1}$ for any x_{T+1}. For the ease of exposition, we aggregate the functions depending on $y_t - d_t(p_t,\epsilon_t)$ and purely on p_t, respectively, and define

$$G_t(y_t - d_t(p_t,\epsilon_t)) = -\mathcal{C}(y_t - d_t(p_t,\epsilon_t)) + \beta V_{t+1}(y_t - d_t(p_t,\epsilon_t)), \tag{6}$$

$$R_t(p_t,\epsilon_t) = (p_t + \beta M) d_t(p_t,\epsilon_t). \tag{7}$$

The equivalent transformation of dynamic recursion is critical in deriving sufficient conditions for the optimality of a BSLP policy in the backorder model. Without this transformation, it would be difficult to investigate the joint concavity of $\widehat{U}_t(p_t,y_t)$ based on the original dynamic recursion in (2).

We then present the sufficient condition for the submodularity of $U_t(p_t,y_t)$ in (5) and also some functional properties of $V'_{t+1}(\cdot)$ and $G'_t(\cdot)$.

LEMMA 2. *Suppose that $V_{t+1}(\cdot)$ is continuously differentiable and concave, then*

(1) $U_t(p_t,y_t)$ is submodular in (p_t,y_t) and hence the smallest optimal price given order-up-to level, denoted by $p_t^*(y_t) = \min \arg\max_{\underline{p} \leq p_t \leq \overline{p}} U_t(p_t,y_t)$ is decreasing in y_t;

(2) $V'_{t+1}(\cdot) \le M$ for $t=0,\cdots,T-1$, and $G'_t(\cdot) \le 0$ for $M \le -b/\beta$ and $t=1,\cdots,T$.

Lemma 2 indicates that the submodularity of the value function $U_t(p_t, y_t)$ under the backorder model holds as long as $V_{t+1}(\cdot)$ is concave. It also reveals a functional property of $V_{t+1}(\cdot)$, i.e., $V'_{t+1}(\cdot) \le M$, which further implies that $\widehat{V}'_{t+1}(\cdot) \le 0$ as $V_{t+1}(x_{t+1}) = \widehat{V}_{t+1}(x_{t+1}) + Mx_{t+1}$. The monotonicity of $\widehat{V}_{t+1}(x_{t+1})$ is consistent with our intuition as a larger x_{t+1} may decrease the selling price p_{t+1} and further hurt the profit $\widehat{V}_{t+1}(x_{t+1})$.

Lemma 2 also reveals our motivation of applying the equivalent transformation for the dynamic recursion of the backorder model as in (4). We make the transformation in order to find the condition under which the function $G_t(\cdot)$ is monotone. The monotonicity of $G_t(\cdot)$ plays a critical role in ensuring the joint concavity of $U_t(p_t, y_t)$ in the backorder model. Notice that, without the equivalent transformation, the first order derivative of $G_t(x_{t+1})$ can either be positive or negative for different x_{t+1} based on the original dynamic recursion in (2).

Note that, if $p_t < b$, the firm is better off by rejecting the demand when there is no inventory on hand to avoid the additional cost $b - p_t$. In this case, a lost-sales model may be more appropriate. Therefore, in this backorder model, we consider $p_t \ge b$ or equivalently $\underline{p} \ge b$.

Based on Lemma 2 and the fact that $\underline{p} \ge b$ in the backorder model, we then show in the following theorem that $V_t(x_t)$ is concave for $t = 1, \cdots, T$ and a BSLP policy is optimal under appropriate conditions. Note that, the optimality of a BSLP policy is implied by the joint concavity and the submodularity of $U_t(p_t, y_t)$.

THEOREM 1. *Suppose $M = -b/\beta$ and $\underline{p} \ge b$. Under the condition that $d_t(p_t, \epsilon_t)$ is strictly decreasing in ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the USDP for all $t = 1, \cdots, T$,*

(1) $V_t(x_t)$ is continuously differentiable and $U_t(p_t, y_t)$ is twice continuously differentiable.

(2) $V_t(x_t)$ is concave in x_t, $U_t(p_t, y_t)$ is submodular and jointly concave in (p_t, y_t).

(3) The optimal policy can be described by a BSLP policy: In each period t, there exist a unique base-stock level $S_t = \arg\max_{y_t} U_t(p^*_t(y_t), y_t)$ and a unique optimal list price $p^*_t(y_t) = \arg\max_{\underline{p} \le p_t \le \overline{p}} U_t(p_t, y_t)$ given order-up-to level y_t such that $p^*_t(y_t)$ is decreasing in y_t; if $x_t < S_t$, it is optimal to order the inventory level up to S_t and set price $p^*_t(S_t)$; otherwise, it is optimal not to order and set price $p^*_t(x_t)$.

REMARK 2. In Theorem 1, the condition that $d_t(p_t, \epsilon_t)$ is strictly decreasing in ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the USDP can be equivalently claimed as that $d_t(p_t, \epsilon_t)$ is strictly increasing in any realization of ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the LSDP since Assumption 3 is made without loss of generality for the monotonicity of demand functions in the realizations of random noises.

To derive the sufficient condition for the optimality of a BSLP policy in the backorder model, we set $M=-b/\beta$ in this theorem due to the following reason. The monotonicity of $G_t(\cdot)$ in (6) and the concavity of $R_t(p_t,\epsilon_t)$ in (7) on p_t are two sufficient conditions that lead to the joint concavity of $U_t(p_t,y_t)$. As in Lemma 2, the monotonicity of $G_t(\cdot)$ can be guaranteed when $M\le -b/\beta$, while the concavity of $R_t(p_t,\epsilon_t)$ can be guaranteed when $p_t+\beta M\ge 0$, i.e., $M\ge -p_t/\beta$. As $p_t\ge \underline{p}\ge b$, $-b/\beta$ is an appropriate value of M such that $-p_t/\beta\le M\le -b/\beta$ in the backorder model. Of course, we can also set $M=-z/\beta$ for any $z\in[b,\underline{p}]$.

Recall that if $f(x,\epsilon)$ is stochastically increasing (decreasing) in x, then it must have the LSDP (USDP) (see Remark 1). Hence, if $d_{t,p}(p_t,\epsilon_t)$ is stochastically increasing (decreasing) in p_t, then $d_{t,p}(p_t,\epsilon_t)$ has the LSDP (USDP). Essentially, if ϵ_t denotes the size of the potential customer base and the sensitivity of demand to price is stochastically increasing in price, i.e., price has a less negative effect on demand as it increases, then the demand sensitivity in price has the LSDP. The notion of stochastic increasing is common in the operations management/research literature.

In addition to the stochastically increasing/decreasing functions, most of the existing demand models have the USDP/LSDP. When $d_t(p_t,\epsilon_t)$ is concave in p_t for any realization of ϵ_t, $d_{t,p}(p_t,\epsilon_t)$ must have the USDP/LSDP. Some demand functions have the USDP/LSDP even when they are not concave in p_t. We list some examples in Table 1 for illustration.

Table 1 **Demand models that have the USDP/LSDP.**

Demand functions that are concave in price:
$d_t(p_t,\epsilon_t)=a-bp_t+\epsilon_t$ (Mills 1959, Petruzzi and Dada 1999);
$d_t(p_t,\epsilon_t)=(\alpha-\beta p_t)^{\gamma}\epsilon_t$, where $\epsilon_t>0$ and $0<\gamma<1$;
$d_t(p_t,\epsilon_t)=\log(\epsilon_t-bp_t)$ (Kocabiyikoğlu and Popescu 2011);
$d_t(p_t,\epsilon_t)=\ln[(a-bp_t)^{\epsilon_t}]$, where $\epsilon_t>0$ (Chen et al. 2006).
Demand functions that are not concave in price:
$d_t(p_t,\epsilon_t)=\frac{p_t^{\epsilon_t+1}}{(\epsilon_t+1)^2}-\frac{p_t^{\epsilon_t+1}}{\epsilon_t+1}\ln(p_t)$, where $\epsilon_t\sim U(-1,0)$ and $p_t\ge 1$;
$d_t(p_t,\epsilon_t)=\epsilon_t\ln(p_t)-p_t^2$, where $\epsilon_t\sim U(-1,0)$ and $p_t\in[\frac{1}{2},1)$.

5.2. The Lost-Sales Model

In the lost-sales model, in each period, any unfilled demand is lost immediately and leftover inventory is carried over to the next period. Given the order-up-to level y_t and the list price p_t in period t, the state at the beginning of period $t+1$ transits to

$$x_{t+1}=[y_t-d_t(p_t,\epsilon_t)]^+.$$

There are a unit holding cost h for leftover inventory and a discount factor $\beta\in[0,1]$. Different from the traditional inventory models without explicitly stating selling prices, under our setting, we do not charge additional lost-sales cost since the lost opportunity cost has been reflected in the revenue function $p_t \min\{y_t, d_t(p_t, \epsilon_t)\}$, which is also adopted by Song et al. (2009). Specifically, the lost opportunity caused by the inventory shortage is $p_t d_t(p_t, \epsilon_t) - p_t y_t$ for $y_t < d_t(p_t, \epsilon_t)$. Then, for any (p_t, y_t), the single-period inventory cost incurred in period t is denoted by $h[y_t - d_t(p_t, \epsilon_t)]^+$.

Let $\widehat{V}_t(\cdot)$ be the expected optimal discounted profit function from period t and onward. The dynamic recursion for this problem is:

$$\widehat{V}_t(x_t) = \max_{\underline{p} \le p_t \le \overline{p},\, y_t \ge x_t} \widehat{U}_t(p_t, y_t) \tag{8}$$

where the objective function $\widehat{U}_t(p_t, y_t)$ is

$$\begin{aligned}\widehat{U}_t(p_t, y_t) &= \int_{\underline{\epsilon}}^{\overline{\epsilon}} \left[p_t \min\{d_t(p_t, \zeta), y_t\} - h[y_t - d_t(p_t, \zeta)]^+ + \beta \widehat{V}_{t+1}([y_t - d_t(p_t, \zeta)]^+)\right] w_t(\zeta) d\zeta \\ &= \int_{\underline{\epsilon}}^{\overline{\epsilon}} \left[p_t y_t - (p_t + h)[y_t - d_t(p_t, \zeta)]^+ + \beta \widehat{V}_{t+1}([y_t - d_t(p_t, \zeta)]^+)\right] w_t(\zeta) d\zeta \end{aligned} \tag{9}$$

for $t = 1, \cdots, T$, and $\widehat{V}_{T+1}(x_{T+1}) \equiv \alpha x_{T+1}$ for all x_{T+1}, where α is the unit salvage cost for the leftover inventory at the end of the selling horizon.

For the ease of exposition, we aggregate the functions depending on $[y_t - d_t(p_t, \epsilon_t)]^+$ and define

$$G_t([y_t - d_t(p_t, \epsilon_t)]^+) = -h[y_t - d_t(p_t, \epsilon_t)]^+ + \beta \widehat{V}_{t+1}([y_t - d_t(p_t, \epsilon_t)]^+). \tag{10}$$

We also denote by $Q_t(p_t, y_t)$ the single-period expected profit, i.e.,

$$Q_t(p_t, y_t) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \left[p_t y_t - (p_t + h)[y_t - d_t(p_t, \zeta)]^+\right] w_t(\zeta) d\zeta. \tag{11}$$

We then provide the sufficient conditions for the optimality of a BSLP policy in the lost-sales model. We adopt the similar analytical approach as in the backorder model. However, in the lost-sales model, the demand is censored by the inventory level, which makes the problem more challenging than the backorder model. For example, $p_t \min\{d_t(p_t, \epsilon_t), y_t\}$ in general is neither concave nor submodular in (p_t, y_t). Moreover, analyzing the Hessian matrix is challenging in the lost-sales model as we are dealing with a multi-period setting.

We tackle the joint concavity of the profit function $\widehat{U}_t(p_t, y_t)$ from period t and onward in the following lemma. The joint concavity is shown by induction based on the negative semi-definite Hessian matrix of $\widehat{U}_t(p_t, y_t)$.

Lemma 3. *Suppose that $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable and concave. The function $\widehat{U}_t(p_t, y_t)$ is jointly concave in (p_t, y_t) if the following conditions hold:*

(i) $d_t(p_t, \epsilon_t)$ is strictly decreasing in any realization of ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the USDP;

(ii) The single-period expected profit function $Q_t(p_t, y_t)$ is submodular in (p_t, y_t).

Based on Lemma 3, we then provide sufficient conditions in the following theorem for the optimality of a BSLP policy in the lost-sales model. Note that, to show the optimality of a BSLP policy, it is sufficient to show that $\widehat{U}_t(p_t, y_t)$ is jointly concave and submodular in (p_t, y_t) through an inductive proof.

THEOREM 2. *Under the following conditions:*

(i) $d_t(p_t, \epsilon_t)$ is strictly decreasing in any realization of ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the USDP;

(ii) the single-period expected profit function $Q_t(p_t, y_t)$ is submodular in (p_t, y_t);

for $t = 1, \cdots, T$, we have the following results:

(1) $\widehat{V}_t(x_t)$ is continuously differentiable and $\widehat{U}_t(p_t, y_t)$ is twice continuously differentiable.

(2) $\widehat{V}_t(x_t)$ is concave in x_t, $\widehat{U}_t(p_t, y_t)$ is submodular and jointly concave in (p_t, y_t);

(3) The optimal policy can be described by a BSLP policy: in each period t, there exist a unique base-stock level $S_t = \arg\max_{y_t} \widehat{U}_t(p_t^*(y_t), y_t)$, and a unique optimal list price $p_t^*(y_t) = \arg\max_{\underline{p} \le p_t \le \overline{p}} \widehat{U}_t(p_t, y_t)$ given order-up-to level y_t, such that $p_t^*(y_t)$ is decreasing in y_t; if the initial inventory level $x_t < S_t$, it is optimal to order the inventory level up to S_t and set price $p_t^*(S_t)$; otherwise, it is optimal not to order and set price $p_t^*(x_t)$.

Two sufficient conditions are required for the optimality of a BSLP policy in the multi-period setting. Similar to the sufficient condition in Theorem 1 for the backorder model, the condition (i) in Theorem 2 can be equivalently claimed as that $d_t(p_t, \epsilon_t)$ is strictly increasing in any realization of ϵ_t and $d_{t,p}(p_t, \epsilon_t)$ has the LSDP. The condition (ii) is required because in general we cannot expect the submodularity of $\widehat{U}_t(p_t, y_t)$ when $Q_t(p_t, y_t)$ is not submodular, even if $\widehat{V}_t(\cdot)$ is concave. We show in the following proposition the sufficient and necessary condition for the submodularity of $Q_t(p_t, y_t)$.

PROPOSITION 2. *Under Assumption 3, the single-period expected profit function $Q_t(p_t, y_t)$ in (11) is submodular in (p_t, y_t) if and only if*

$$\Delta(p_t, y_t) = -\frac{(p_t + h)\partial \bar{F}_t(p_t, y_t)/\partial p_t}{\bar{F}_t(p_t, y_t)} = \frac{(p_t + h)\partial F_t(p_t, y_t)/\partial p_t}{\bar{F}_t(p_t, y_t)} \ge 1, \tag{12}$$

where $\bar{F}_t(p_t, y_t) = P(d_t(p_t, \epsilon_t) > y_t)$ is the lost-sales rate.

Proposition 2 indicates that the inequality (12) is the sufficient and necessary condition to ensure the submodularity of the single-period expected profit function $Q_t(p_t, y_t)$.

In fact, when $h = 0$, Proposition 2 is essentially the same as the Proposition 1 in Kocabiyikoğlu and Popescu (2011). $\Delta(p_t, y_t)$ with $h = 0$ is the lost sales rate (LSR) elasticity defined in

Kocabiyikoğlu and Popescu (2011). Specifically, the LSR elasticity is the percentage change in the rate of lost sales w.r.t the percentage change in price for a given quantity. Hence, the submodularity of $Q_t(p_t, y_t)$ can be guaranteed if and only if the LSR elasticity is no smaller than 1. They also show that the inequality (12) is the sufficient condition to ensure the joint concavity of the single-period expected profit function with $h = 0$. In other words, the joint concavity of the profit function can be guaranteed by its submodularity in the single-period setting. However, in our multi-period setting, the submodularity of the single-period profit function alone cannot ensure the joint concavity of the profit-to-go function.

Similar to Kocabiyikoğlu and Popescu (2011), under certain conditions, the monotonicity of $\Delta(p_t, y_t)$ w.r.t p_t, y_t holds. Let IFR and DFR denote increasing failure rate and decreasing failure rate, respectively. We present the following results.

LEMMA 4. *If $d_t(p_t, \epsilon_t)$ is strictly increasing (decreasing) in any realization of ϵ_t, ϵ_t has IFR (DFR), and $d_{t,\zeta}(p_t, \bar{\zeta}_t)$ is decreasing (increasing) in p_t, then $\Delta(p_t, y_t)$ is increasing in y_t; if further $d_{t,p}(p_t, \bar{\zeta}_t)/d_{t,\zeta}(p_t, \bar{\zeta}_t)$ is decreasing (increasing) in p_t, then $\Delta(p_t, y_t)$ is also increasing in p_t.*

When $d_t(p_t, \epsilon_t)$ is increasing in any realization of ϵ_t, it is easy to verify that our sufficient condition that $d_{t,\zeta}(p_t, \bar{\zeta}_t)$ is decreasing in p_t can be guaranteed by the sufficient condition in Proposition 4 of Kocabiyikoğlu and Popescu (2011) for the monotone increasing property of the LSR elasticity w.r.t y_t. In this sense, we provide more general sufficient conditions than Kocabiyikoğlu and Popescu (2011). For the monotone increasing property of the LSR elasticity w.r.t p_t, we provide a set of sufficient conditions that complements those in Kocabiyikoğlu and Popescu (2011).

Suppose that, in each period t, there is a nonnegative minimum demand quantity denoted by $\underline{d}_t$. Then, under the optimal policy, we must have $S_t \geq \underline{d}_t$, i.e., the optimal base stock level should be more than the minimum demand. With the monotone increasing property of $\Delta(p_t, y_t)$ w.r.t y_t, p_t, the inequality (12) can be guaranteed if

$$-\frac{(\underline{p} + h)\partial \bar{F}_t(\underline{p}, \underline{d}_t)/\partial \underline{p}}{\bar{F}_t(\underline{p}, \underline{d}_t)} \geq 1.$$

We then illustrate some demand functions, as well as their corresponding conditions, under which the two sufficient conditions in Theorem 2 for the optimality of a BSLP policy can be guaranteed. Recall that $p_t \in [\underline{p}, \overline{p}]$, $\epsilon_t \in [\underline{\epsilon}, \overline{\epsilon}]$, W_t and w_t are the distribution function and the density function of random noise ϵ_t, respectively.

COROLLARY 1. *Suppose that W_t has the IFR property and there is a nonnegative minimum demand denoted by $\underline{d}_t$ in each period t. The two sufficient conditions in Theorem 2 can be guaranteed by the following demand functions and their corresponding conditions:*

(1) Under an additive demand function $d_t(p_t,\epsilon_t)=\mu_t(p_t)+\epsilon_t$, the two conditions hold if $\mu_t''(p_t)\leq 0$ and $-(\underline{p}+h)w_t(\underline{d}_t-\mu_t(\underline{p}))\mu_t'(\underline{p})\geq \bar{W}_t(\underline{d}_t-\mu_t(\underline{p}))$;

(2) Under a multiplicative demand function $d_t(p_t,\epsilon_t)=\sigma_t(p_t)\epsilon_t$ for $\underline{\epsilon}\geq 0$ and $\sigma_t(p_t)\geq 0$, the two conditions hold if $\sigma_t''(p_t)\leq 0$ and $-(\underline{p}+h)w_t\left(\frac{\underline{d}_t}{\sigma_t(\underline{p})}\right)\underline{d}_t\sigma_t'(\underline{p})\geq \bar{W}_t\left(\frac{\underline{d}_t}{\sigma_t(\underline{p})}\right)\sigma_t^2(\underline{p})$;

(3) Under a generalized additive demand function $d_t(p_t,\epsilon_t)=\mu_t(p_t)+\sigma_t(p_t)\epsilon_t$ for $\underline{\epsilon}\geq 0$, $\mu_t(p_t)$ and $\sigma_t(p_t)$ are concavely decreasing and nonnegative, the two conditions hold if $-(\underline{p}+h)w_t\left(\frac{\underline{d}_t-\mu_t(\underline{p})}{\sigma_t(\underline{p})}\right)\left(\mu_t'(\underline{p})\sigma_t(\overline{p})+(\underline{d}_t-\mu_t(\underline{p}))\sigma_t'(\underline{p})\right)\geq \bar{W}_t\left(\frac{\underline{d}_t-\mu_t(\underline{p})}{\sigma_t(\underline{p})}\right)\sigma_t^2(\underline{p})$;

(4) Under a logarithmic demand function $d_t(p_t,\epsilon_t)=\ln(bp_t+\epsilon_t)$ for $b\leq 0$ and $\underline{\epsilon}\geq 0$, the two conditions hold if $-b(\underline{p}+h)w_t(e^{\underline{d}_t}-b\underline{p})\geq \bar{W}_t(e^{\underline{d}_t}-b\underline{p})$.

Corollary 1 follows directly from the condition in (12) and the IFR property. The IFR property holds for most well-known distributions, e.g., uniform, normal, truncated-normal, logistic, log-normal, exponential, Laplace, Weibull distributions, etc.

In Corollary 1, we only illustrate the demand functions that are increasing in the random noise ϵ_t and require that the random noise has the IFR property. When demand functions are decreasing in ϵ_t, with the DFR property of the random noise, we can also find some simple conditions under which the two conditions required in Theorem 2 are guaranteed. The conditions may be further simplified if the demand functions are specifically identified. For example, under the complex demand function $d_t(p_t,\epsilon_t)=\epsilon_t\ln(p_t)-p_t^2$ for $\epsilon_t\sim U(-1,0)$ and $p_t\in[\frac{1}{2},1)$ (its $d_{t,p}(p_t,\epsilon_t)$ has the USDP as discussed in Section 5.1), the two sufficient conditions in Theorem 2 can be simply guaranteed as long as $h\geq 2\ln^2(2)-0.5$. The analytical approach is the same as those in Corollary 1. Through the above discussion, we show that our sufficient conditions in Theorem 2 are not restrictive. Under some specific demand functions, we can even ensure the optimality of a BSLP policy in the lost-sales model with simple conditions.

5.3. Discussions

In this section, we present some discussions for the results in the backorder model and the lost-sales model sequentially. For the backorder model, the strict monotonicity of the demand function w.r.t the realizations of random noises plays an important role in deriving its results in Theorem 1. In Federgruen and Heching (1999), to ensure the joint concavity and submodularity of the value function for the backorder model, they require that (1) $d_t(p_t,\epsilon_t)$ is decreasing and concave in p_t for any realization of ϵ_t and (2) $\int_{-\infty}^{+\infty}\mathcal{C}(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta$ is jointly convex in (p_t,y_t). Validating the condition (2) is non-trivial as mentioned in Federgruen and Heching (1999) and Feng et al. (2013). In fact, it cannot be guaranteed even if $d_{t,p}(p_t,\epsilon_t)$ has the USDP/LSDP. However, by utilizing the strict monotonicity of $d_t(p_t, \epsilon_t)$ w.r.t any realization of ϵ_t, Theorem 1 indicates that the condition (1) alone can indeed guarantee the joint concavity and submodularity of the value function. This fact implies the value of exploiting the monotonicity of the random noises in a demand function.

Table 2 Sufficient conditions under the generalized additive demand function $d_t(p_t,\epsilon_t) = \mu_t(p_t) + \sigma_t(p_t)\epsilon_t$.

	Demand Function	Sufficient Conditions
Feng et al. (2013)	$\mu_t(p_t) > 0$ $\sigma_t(p_t) > 0$ $\mathbb{E}[\epsilon_t] = 0$, $\text{Var}[\epsilon_t] = 1$	$\mu_t'(p_t) \le 0$ $\sigma_t'(p_t) < 0$ $\mu_t'(p_t)\frac{\sigma_t''(p_t)}{\sigma_t'(p_t)} \ge \mu_t''(p_t)$ $\mu_t(p_t)\frac{\sigma_t''(p_t)}{\sigma_t'(p_t)} \ge 2\mu_t'(p_t)$ $\frac{\mu_t'(p_t)}{\mu_t(p_t)} \le \frac{\sigma_t'(p_t)}{\sigma_t(p_t)}$
Our results	$\mu_t(p_t) > 0$ $\sigma_t(p_t) > 0$ $\epsilon_t \in [0, \bar{\epsilon}]$	$\mu_t'(p_t) \le 0$ $\sigma_t'(p_t) < 0$ $\mu_t''(p_t) \le 0$ $\sigma_t''(p_t) < 0$

In the backorder model, Feng et al. (2013) provide a set of conditions to ensure the optimality of a BSLP policy under the generalized additive demand function $d_t(p_t, \epsilon_t) = \mu_t(p_t) + \sigma_t(p_t)\epsilon_t$, which is a widely applied demand function in the existing literature. Table 2 lists the sufficient conditions given by Feng et al. (2013) and our results. The conditions provided by Feng et al. (2013) are quite complex and it is not easy to find the functions of σ_t, μ_t that satisfy their conditions. Compared with Feng et. al (2013), we adopt a different approach and our results complement theirs. Moreover, our results are not restricted by the generalized additive demand function.

For the lost-sales model, Kocabiyikoğlu and Popescu (2011) have investigated the optimality of a BSLP policy in a *single-period setting* with general price-dependent demands. They show that it is challenging to derive the sufficient conditions even in the single-period problem. However, as stated by Feng et al. (2013), the approach used by Kocabiyikoğlu and Popescu (2011) is not helpful to tackle multi-period models as the technique needed for multi-period models can be fundamentally different from that for single-period problems. This is because that, in general, $\mathbb{E}\left[\widehat{V}_{t+1}([y_t - d_t(p_t,\epsilon_t)]^+)\right]$ may not be concave in (p_t, y_t) even if $d_t(p_t, \epsilon_t)$ is concave in p_t and $\widehat{V}_{t+1}$ is concave, i.e., the composite of a concave function with a convex function may not be concave. The fact may explain why in the existing inventory-pricing literature the analysis on the lost-sales model is very limited.

In the single-period setting, Kocabiyikoğlu and Popescu (2011) provide the following sufficient conditions for the optimality of a BSLP policy when $h = 0$: (a) $d_t(p_t, \epsilon_t)$ is decreasing in p_t and increasing in any realizations of ϵ_t, (b) the riskless unconstrained revenue $p_t d_t(p_t, \epsilon_t)$ is strictly concave in p_t for any realizations of ϵ_t, and (c) the single-period profit function $Q_t(p_t, y_t)$ is submodular. The condition (a) in Kocabiyikoğlu and Popescu (2011) is consistent with Assumption 1, and the condition (c) is the same with our condition (ii). However, unlike the condition (b) in the single-period setting, we require that $d_{t,p}(p_t, \epsilon_t)$ has the USDP/LSDP for the optimality of a BSLP policy in the multi-period setting. The USDP/LSDP of $d_{t,p}(p_t, \epsilon_t)$ can guarantee the concavity of $p_t \mathbb{E}_{\epsilon_t}[d_t(p_t, \epsilon_t)]$ but not the concavity of $p_t d_t(p_t, \epsilon_t)$ in p_t for any ϵ_t since $d_t(p_t, \epsilon_t)$ can be non-concave in p_t as we have discussed in Section 5.1. In this sense, our sufficient condition can partially relax the condition (b) in Kocabiyikoğlu and Popescu (2011).

6. Extensions

In this section, to show the applications of USDP/LSDP, we analyze inventory management problems with inventory-dependent demands and quality-dependent demands. Similar to the basic model with price-dependent demands, the USDP/LSDP serves as a more general condition, compared with the restrictive sufficient conditions in the existing literature, that leads to the (joint) concavity of the value functions for the two problems discussed below.

6.1. Inventory Management with Inventory-Dependent Demands

There is a widely recognized phenomenon in marketing that the demand of many retail items is affected by the amount of inventory displayed, e.g., the scarcity effect. We refer to such kind of demand as the inventory-dependent demand. The inventory-dependent demand has been considered by, e.g., Yang and Zhang (2014) and Smith and Agrawal (2017) in the existing literature. Sapra et al. (2010) consider inventory management with a multiplicative form of inventory-dependent demand under the backorder setting. Following Sapra et al. (2010), we assume that the demand is decreasing in the ending inventory in the previous period but we consider a general demand model. We denote by $d_t(x_t, \epsilon_t)$ the inventory-dependent demand, where x_t is the initial inventory level at the beginning of period t. Similar to the basic model, we normalize the unit ordering cost to 0. To focus on the inventory problem, we assume that there is a fixed selling price p for each unit.

We denote by $V_t(x_t)$ the expected optimal discounted cost function from period t and onward, and $d_{t,x}(x_t, \epsilon_t)$ the first-order derivative of $d_t(x_t, \epsilon_t)$ w.r.t x_t. The dynamic recursions can be found in Appendix A. For the backorder model, we have the following results.

PROPOSITION 3 **(The Backorder Model)**. *Let $M = -b/\beta$. For $t = 1, \cdots, T$, if $d_t(x_t, \epsilon_t)$ is decreasing in x_t, $d_t(x_t, \epsilon_t)$ is strictly decreasing (increasing) in any realization of ϵ_t, and $d_{t,x}(x_t, \epsilon_t)$ has the USDP (LSDP), then we have the following results:*

(1) $V_t(x_t)$ is concave and $V_t'(x_t) \leq -b/\beta$;

(2) An inventory-dependent base-stock policy is optimal: there exists $y_t^*(x_t)$ such that if $x_t < y_t^*(x_t)$, then it is optimal to order up to $y_t^*(x_t)$; otherwise, it is optimal to order nothing;

(3) $y_t^*(x_t)$ is decreasing in x_t.

Similarly, for the lost-sales model, we have the following results.

Proposition 4 **(The Lost-Sales Model)**. *For $t = 1, \cdots, T$, if $d_t(x_t, \epsilon_t)$ is decreasing in x_t, $d_t(x_t, \epsilon_t)$ is strictly decreasing (increasing) in any realization of ϵ_t, and $d_{t,x}(x_t, \epsilon_t)$ has the USDP (LSDP), then we have the following results:*

(1) $V_t(x_t)$ is concave and $V_t'(x_t) \leq 0$ for any $x_t \geq 0$;

(2) An inventory-dependent base-stock policy is optimal: there exists $y_t^*(x_t)$ such that if $x_t < y_t^*(x_t)$, then it is optimal to order up to $y_t^*(x_t)$; otherwise, it is optimal to order nothing;

(3) $y_t^*(x_t)$ is decreasing in x_t.

6.2. Inventory Management with Quality-Dependent Demands

In practice, firms may order semi-products and then make the end-products once the demands are realized, e.g., the postponement strategy of mass customization, so that they can control the quality level of the end-products. It is well known that the demand is induced by quality in the marketing literature. In each period, the firm first announces the quality level of end-products, denoted by $\theta_t \in [0, 1]$, and then observes the quality-dependent demand $d_t(\theta_t, \epsilon_t)$. The demand $d_t(\theta_t, \epsilon_t)$ is increasing in θ_t in each period t. The firm designs the end-products only after receiving demands and hence the left-over inventory only consists of the semi-products. We denote by $K(\theta_t)$ the total quality positioning cost, e.g., the designing cost, on the quality level θ_t in period t. As $K(\theta_t)$ is the designing cost, it is independent of the order quantity in period t. Following Jing (2017), we assume that $K(\theta_t)$ is increasing and convex in θ_t, i.e., it is more costly to improve the quality level when θ_t is larger. There is a fixed unit selling price p. Similar to the basic model, we normalize the unit ordering cost to 0.

We denote by $V_t(x_t)$ the expected optimal discounted cost function from period t and onward, $U_t(\theta_t, y_t)$ the objective function of the optimization problem as in (4) and $d_{t,\theta}(\theta_t, \epsilon_t)$ the first-order derivative of the demand function w.r.t θ_t. The dynamic recursions can be found in Appendix A. For the backorder model, we have the following results.

Proposition 5 **(The Backorder Model)**. *Let $M = -b/\beta$. For $t = 1, \cdots, T$, if $d_t(\theta_t, \epsilon_t)$ is increasing in θ_t, $d_t(\theta_t, \epsilon_t)$ is strictly decreasing (increasing) in any realization of ϵ_t, and $d_{t,\theta}(\theta_t, \epsilon_t)$ has the USDP (LSDP), then we have the following results:*

(1) $V_t(x_t)$ is concave and $U_t(\theta_t, y_t)$ is supermodular and jointly concave in (θ_t, y_t);

(2) A base-stock list quality level policy is optimal: there exist a unique base-stock level $S_t = \arg\max_{y_t} U_t(\theta_t^*(y_t), y_t)$ *and a unique optimal list quality level* $\theta_t^*(y_t) = \arg\max_{\theta_t \in [0,1]} U_t(\theta_t, y_t)$ *given order-up-to level* y_t *such that* $\theta_t^*(y_t)$ *is increasing in* y_t*; if* $x_t < S_t$*, it is optimal to order the inventory level up to* S_t *and set quality level* $\theta_t^*(S_t)$*; otherwise, it is optimal to not to order and set quality level* $\theta_t^*(x_t)$*.*

Similarly, for the lost-sales model, we have the following results.

PROPOSITION 6 **(The Lost-Sales Model)**. *For* $t = 1, \cdots, T$*, if* $d_t(\theta_t, \epsilon_t)$ *is increasing in* θ_t*,* $d_t(\theta_t, \epsilon_t)$ *is strictly decreasing (increasing) in any realization of* ϵ_t*, and* $d_{t,\theta}(\theta_t, \epsilon_t)$ *has the USDP (LSDP), then we have the following results:*

(1) $V_t(x_t)$ *is concave and* $V_t'(x_t) \leq 0$ *for any* $x_t \geq 0$*;*

(2) A base-stock list quality level policy is optimal: there exist a base-stock level S_t *and an optimal list quality level* $\theta_t^*(y_t)$ *given order-up-to level* y_t *such that* $\theta_t^*(y_t)$ *is increasing in* y_t*; if* $x_t < S_t$*, it is optimal to order the inventory level up to* S_t *and set quality level* $\theta_t^*(S_t)$*; otherwise, it is optimal to not to order and set quality level* $\theta_t^*(x_t)$*.*

7. Concluding Remarks

The joint inventory and pricing control has received considerable attention in the literature. For the validation of the optimality of a BSLP policy, the existing literature suffers two limitations: for the backorder setting, it requires unnecessarily restrictive functional forms of demand while the results for the lost-sales setting are very limited, especially for the multi-period setting. This may cast doubts over the applicability of the BSLP policy in practice.

To overcome these two limitations, we utilize the monotonicity of the demand functions in the realizations of random noises to derive structural properties of the value functions. Based on such a monotone property, for the backorder model, we show that a BSLP policy is optimal in each period if demand functions are decreasing in price and strictly decreasing (increasing) in the realizations of random noises, and their sensitivities in price have the USDP (LSDP), where USDP/LSDP is a generalized version of the first-order stochastic dominance. For the lost-sales model, we show a BSLP policy is optimal if demand functions are decreasing in price and strictly decreasing (increasing) in the realizations of random noises, their sensitivities in price have the USDP (LSDP), and the single-period expected profit function is submodular in price and inventory level.

Our results can be generalized to systems with Markov modulated demand under similar conditions. For Markov modulated demand, under similar conditions, we can show that a state-dependent

BSLP policy is optimal, where the *state* refers to the Markov modulating state. Through the above discussion, we show that the optimality of a BSLP policy is robust to more general business environments than what we previously thought. We also apply the USDP/LSDP in other operations management models where demand could depend on other factors, e.g., quality and inventory.

In our settings, we assume that all unsatisfied demands are either backordered or lost immediately. It is interesting to generalize our results to the cases with partial backorder and partial lost-sales. This may be possible since we show that if demand functions are decreasing and concave in price and are strictly monotone in the realizations of random noises, then a BSLP policy is optimal for either the backorder setting or the lost-sales setting.

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References

Bernstein, F., Li, Y., Shang, K. 2016. A simple heuristic for joint inventory and pricing models with lead time and backorders. *Management Science*, **62**(8): 2358-2373.

Chan, L. M., Shen, Z. M., Simchi-Levi, D., J. L. Swann. 2004. Coordination of pricing and inventory decisions: A survey and classification. In *Handbook of Quantitative Supply Chain Analysis*, Springer, US, 335-392.

Chen, Y. F., Ray, S., Y. Song. 2006. Optimal pricing and inventory control policy in periodic-review systems with fixed ordering cost and lost sales. *Naval Research Logistics*, **53**(2): 117-136.

Chen, X., D. Simchi-Levi. 2004a. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The finite horizon case. *Operations Research*, **52**(6): 887-896.

Chen, X., D. Simchi-Levi. 2004b. Coordinating inventory control and pricing strategies with random demand and fixed ordering cost: The infinite horizon case. *Mathematics of Operations Research*, **29**(3): 698-723.

Chen, X., D. Simchi-Levi. 2012. Pricing and inventory management. Ö., Özer and R. Phillips (Eds). *The Oxford Handbook of Pricing Management*, Oxford University Press, United Kingdom, 784-824.

Chen, H., Z. Zhang. 2014. Joint inventory and pricing control with general additive demand. *Operations Research*, **62**(6): 1335-1343.

Chen, X., Zhang, Y., S. X. Zhou. 2010. Preservation of quasi-K-concavity and its applications. *Operations Research*, **58**(4): 1012-1016.

Cheng, T. C. E. 1984. An integrated product pricing and batch sizing model with stochastic demand. *Engineering Costs and Production Economics*, **8**(1): 27-31.

Elmaghraby, W., P. Keskinocak. 2003. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science*, **49**(10): 1287-1309.

Federgruen, A., A. Heching. 1999. Combined pricing and inventory control under uncertainty. *Operations Research*, **47**(3): 454-475.

Feng, Q., Luo, S., D. Zhang. 2013. Dynamic inventory-pricing control under backorder: Demand estimation and policy optimization. *Manufacturing and Service Operations Management*, **16**(1): 149-160.

Huh, W. T., G. Janakiraman. 2008. (s, S) optimality in joint inventory-pricing control: An alternate approach. *Operations Research*, **56**(3): 783-790.

Jing, B. 2017. Behavior-Based Pricing, Production Efficiency, and Quality Differentiation. *Management Science*, **63**(7): 2365-2376.

Karlin, S., C. R. Carr. 1962. Prices and optimal inventory policy. K. J., Arrow, S., Karlin, and H., Scarf (Eds.). *Studies in Applied Probability and Management Science*, Stanford University Press. Stanford, CA, 159-172.

Kocabiyikoğlu, A., I. Popescu. 2011. An elasticity approach to the newsvendor with price-sensitive demand. *Operations Research*, **59**(2): 301-312.

Lau, A. H. L., H. S. Lau. 1988. The newsboy problem with price-dependent demand distribution. *IIE transactions*, **20**(2): 168-175.

Lu, Y., Song, M., Yang, Y. 2016. Joint inventory and pricing coordination with incomplete demand information. *Production and Operations Management*, **25**(4): 701-718.

Lu, Y., D. Simchi-Levi. 2013. On the unimodality of the profit function of the pricing newsvendor. *Production and Operations Management*, **22**(3): 615-625.

Mills, E. S. 1959. Uncertainty and price theory. *The Quarterly Journal of Economics*, **73**(1): 116-130.

Mills, E. S. 1962. *Price, Output and Inventory Policy.* John Wiley, New York.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Operations Research*, **47**(2): 183-194.

Polatoglu, L. H. 1991. Optimal order quantity and pricing decisions in single-period inventory systems. *International Journal of Production Economics*, **23**(1): 175-185.

Polatoglu, H., I. Sahin. 2000. Optimal procurement policies under price-dependent demand. *International Journal of Production Economics*, **65**(2): 141-171.

Rudin, W. 1991. *Functional analysis. International series in pure and applied mathematics*, McGraw-Hill, Inc., New York.

Sapra, A., Truong, V. A., R. Q. Zhang. 2010. How much demand should be fulfilled? *Operations Research*, **58**(3): 719-733.

Smith, S. A., N. Agrawal. 2017. Optimal Markdown Pricing and Inventory Allocation for Retail Chains with Inventory Dependent Demand. *Manufacturing and Service Operations Management*, **19**(2): 290-304.

Song, Y., Ray, S., T. Boyaci. 2009. Optimal dynamic joint inventory-pricing control for multiplicative demand with fixed order costs and lost sales. *Operations Research*, **57**(1): 245-250.

Shaked, M., J. G. Shanthikumar. 1994. *Stochastic orders and their applications.* Academic Press.

Thowsen, G. T. 1975. A dynamic, nonstationary inventory problem for a price/quantity setting firm. *Naval Research Logistics Quarterly*, **22**(3): 461-476.

Whitin, T. M. 1955. Inventory control and price theory. *Management Science*, **2**(1): 61-68.

Yang, N., R. Zhang. 2014. Dynamic pricing and inventory management under inventory-dependent demand. *Operations Research*, **62**(5): 1077-1094.

Yano, C. A., S. M. Gilbert. 2004. Coordinated pricing and production/procurement decisions: A Review. A., Chakravarty, and J., Eliashberg (Eds.). *Managing Business Interfaces: Marketing, Engineering and Manufacturing Perspectives.* Kluwer Academic Publishers. Springer, US, 65-103.

Yao, L., Chen, Y. F., H. Yan. 2006. The newsvendor problem with pricing: Extensions. *International Journal of Management Science and Engineering Management*, **1**(1): 3-16.

Young, L. 1978. Price, inventory and the structure of uncertain demand. *New Zealand Operations Research*, **6**(2): 157-177.

Zabel, E. 1970. Monopoly and uncertainty. *The Review of Economic Studies*, **37**(2): 205-219.

Zabel, E. 1972. Multiperiod monopoly under uncertainty. *Journal of Economic Theory*, **5**(3): 524-536.

Appendix A: Main Proofs

Denote by $d_{t,pp}(p_t,\zeta)$, $d_{t,p\zeta}(p_t,\zeta)$, and $d_{t,\zeta\zeta}(p_t,\zeta)$ the second order partial derivatives, and $d_{t,p\zeta\zeta}(p,\zeta)$, $d_{t,pp\zeta}(p,\zeta)$, etc. the third order partial derivatives of $d_t(p_t,\zeta)$.

Proof of Lemma 1.

We first consider that $f(x,\epsilon)$ has the USDP and $g(\zeta)$ is non-positive and decreasing in ζ. In this case, $g(\epsilon)$ can be approximated by a sum of decreasing step functions. Specifically, we can divide the interval $[\mu,\bar{\epsilon}]$ into $N+1$ disjoint intervals by the set of points $\mathcal{N}=\{u_1,\cdots,u_N\}$ such that $u_1<u_2<\cdots<u_N$. Then $g(\epsilon)$ can be approximated as a sum $g(\epsilon)\approx a_0+\sum_{j=1}^N a_j\mathbb{I}_{\{\epsilon\geq u_j\}}$, where a_j for $j=0,1,\cdots,N$ are non-positive and $\sum_{j=0}^i a_j=g(u_i)$ for $i=1,\cdots,N$. If the partition is more dense, it is clear that the approximation is more accurate. The integral $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ can be approximated as

$$\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta\approx F(x,\mu,N)=a_0\int_\mu^{\bar{\epsilon}} f(x,\zeta)\omega(\zeta)d\zeta+\sum_{j=1}^N a_j\int_{u_j}^{\bar{\epsilon}} f(x,\zeta)\omega(\zeta)d\zeta. \tag{13}$$

Since $f(x,\epsilon)$ has the USDP, it follows that given any partition $\mathcal{N}$, the approximation of $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ must be increasing in x since $a_j\leq 0$ for $j=0,1,\cdots,N$. In addition, the value of $F(x,\mu,N)$ decreases in N for any x because $a_j\leq 0$ for $j=0,\cdots,N$. Hence, as $N\to\infty$, the approximation converges to $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ based on the monotone convergence theorem (Yeh 2006). As a result, $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ must be increasing in x. Hence, for any $g(\zeta)$ that is non-positive and decreasing in ζ, $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ for any $\mu\in[\underline{\epsilon},\bar{\epsilon}]$ is increasing in x if $f(x,\epsilon)$ has the USDP.

Similarly, if the approximation in (13) is increasing in x for any non-positive and decreasing function $g(\epsilon)$, then we can show that the function $f(x,\epsilon)$ must have the USDP by letting $a_j<0$ for any $j\in\{0,\cdots,N\}$ and $a_i=0$ for $i=0,\cdots,N$ and $i\neq j$. Hence, $f(x,\epsilon)$ has the USDP if $\int_\mu^{\bar{\epsilon}} g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ for any $\mu\in[\underline{\epsilon},\bar{\epsilon}]$ is increasing in x for any $g(\zeta)$ that is non-positive and decreasing in ζ.

We then consider that $f(x,\epsilon)$ has the LSDP and $g(\zeta)$ is non-positive and increasing in ζ. We still divide the interval $[\underline{\epsilon},\mu]$ into $N+1$ disjoint intervals by the set of points $\mathcal{N}=\{u_1,\cdots,u_N\}$ such that $u_1<u_2<\cdots<u_N$. Then, $g(\epsilon)$ can be approximated by a sum of increasing step functions, i.e., $g(\epsilon)\approx\sum_{j=1}^N a_j\mathbb{I}_{\{\epsilon\leq u_j\}}+a_{N+1}$, where $a_j\leq 0$ for $j=1,\cdots,N,N+1$ and $\sum_{j=i}^{N+1} a_j=g(u_i)$ for $i=1,\cdots,N$. Then, the integral $\int_{\underline{\epsilon}}^\mu g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ can be approximated as

$$\int_{\underline{\epsilon}}^\mu g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta\approx\sum_{j=1}^N a_j\int_{\underline{\epsilon}}^{u_j} f(x,\zeta)\omega(\zeta)d\zeta+a_{N+1}\int_{\underline{\epsilon}}^\mu f(x,\zeta)\omega(\zeta)d\zeta. \tag{14}$$

Since $f(x,\epsilon)$ has the LSDP, it follows that given any partition $\mathcal{N}$, the approximation of $\int_{\underline{\epsilon}}^\mu g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ must be increasing in x since $a_j\leq 0$ for $j=1,\cdots,N,N+1$. With the similar argument above, we then can show that for any non-positive and increasing function $g(\epsilon)$, $\int_{\underline{\epsilon}}^\mu g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ is increasing in x if $f(x,\epsilon)$ has the LSDP.

Similarly, we can show that $f(x,\epsilon)$ has the LSDP if $\int_{\underline{\epsilon}}^\mu g(\zeta)f(x,\zeta)\omega(\zeta)d\zeta$ for any $\mu\in[\underline{\epsilon},\bar{\epsilon}]$ is increasing in x for any non-positive and increasing function $g(\epsilon)$ by letting $a_j<0$ for any $j\in\{1,\cdots,N+1\}$ and $a_i=0$ for $i=1,\cdots,N+1$ and $i\neq j$ in (14). ■

Proof of Proposition 1.

The first order derivatives of $F(x,z)=\int_{\underline{\epsilon}}^{\overline{\epsilon}} f(x-\psi(z,\zeta))\omega(\zeta)d\zeta$ are

$$\begin{cases}\partial F(x,z)/\partial x = \int_{\underline{\epsilon}}^{\overline{\epsilon}} f'(x-\psi(z,\zeta))\omega(\zeta)d\zeta,\\ \partial F(x,z)/\partial z = -\int_{\underline{\epsilon}}^{\overline{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z,\zeta)\omega(\zeta)d\zeta.\end{cases}$$

The differentiation operator and the integration operator can be interchanged based on the Leibniz integral rule. Then, its second order partial derivatives are

$$\frac{\partial^2 F(x,z)}{\partial x^2} = \int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\omega(\zeta)d\zeta, \tag{15}$$

$$\frac{\partial^2 F(x,z)}{\partial z^2} = \int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z^2(z,\zeta)\omega(\zeta)d\zeta - \int_{\underline{\epsilon}}^{\overline{\epsilon}} f'(x-\psi(z,\zeta))\psi_{zz}(z,\zeta)\omega(\zeta)d\zeta, \tag{16}$$

$$\frac{\partial^2 F(x,z)}{\partial x \partial z} = -\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z(z,\zeta)\omega(\zeta)d\zeta, \tag{17}$$

where $\psi_z(z,\zeta)$ and $\psi_{zz}(z,\zeta)$ denote the first order derivative and the second order partial derivative of $\psi(z,\zeta)$ w.r.t z.

To ensure the joint concavity of $F(x,z)$, its corresponding Hessian matrix shall be negative semidefinite, i.e.,

$$\frac{\partial^2 F(x,z)}{\partial x^2} \le 0, \tag{18}$$

$$\frac{\partial^2 F(x,z)}{\partial x^2}\frac{\partial^2 F(x,z)}{\partial z^2} \ge \left(\frac{\partial^2 F(x,z)}{\partial x \partial z}\right)^2, \tag{19}$$

$$\frac{\partial^2 F(x,z)}{\partial z^2} \le 0. \tag{20}$$

Note that, if both of the inequalities (18) and (19) are valid, then the inequality (20) must hold since $\left(\frac{\partial^2 F(x,z)}{\partial x \partial z}\right)^2 \ge 0$. Therefore, we only need to show in the following that the inequalities (18) and (19) are guaranteed by the conditions in Proposition 1.

Through the expression in (15), it is easy to observe that the inequality (18) holds under the condition (i) since $f''(\cdot) \le 0$ when $f(\cdot)$ is concave. By plugging in the expressions in (15)-(17), the inequality (19) can be denoted by

$$\begin{aligned}&\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\omega(\zeta)d\zeta \int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z^2(z,\zeta)\omega(\zeta)d\zeta\\ &-\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\omega(\zeta)d\zeta \int_{\underline{\epsilon}}^{\overline{\epsilon}} f'(x-\psi(z,\zeta))\psi_{zz}(z,\zeta)\omega(\zeta)d\zeta\\ \ge& \left(\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z(z,\zeta)\omega(\zeta)d\zeta\right)^2.\end{aligned} \tag{21}$$

Note that, according to the Cauchy-Schwarz inequality (Steele 2004),

$$\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\omega(\zeta)d\zeta \int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z^2(z,\zeta)\omega(\zeta)d\zeta \ge \left(\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\psi_z(z,\zeta)\omega(\zeta)d\zeta\right)^2.$$

In addition, due to the condition (i), i.e., the concavity of $f(\cdot)$, we must have that $\int_{\underline{\epsilon}}^{\overline{\epsilon}} f''(x-\psi(z,\zeta))\omega(\zeta)d\zeta \le 0$. Hence, the inequality (21) holds if

$$\int_{\underline{\epsilon}}^{\overline{\epsilon}} f'(x-\psi(z,\zeta))\psi_{zz}(z,\zeta)\omega(\zeta)d\zeta \ge 0. \tag{22}$$

Note that

$$
\begin{aligned}
&\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z+\delta,\zeta)\omega(\zeta)d\zeta - \int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z,\zeta)\omega(\zeta)d\zeta \\
=&\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\left(\psi_z(z+\delta,\zeta)-\psi_z(z,\zeta)\right)\omega(\zeta)d\zeta \\
=&\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_{zz}(z,\zeta)\delta\omega(\zeta)d\zeta \quad \text{a.s. } \delta\to 0_+.
\end{aligned}
$$

Hence, (22) holds if, for any (x,z), $\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z+\delta,\zeta)\omega(\zeta)d\zeta$ is increasing in δ (by letting $\delta\to 0_+$). Clearly, it also holds if, for any (x,z), $\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z+\delta,\zeta)\omega(\zeta)d\zeta$ is increasing in arbitrary $\delta>0$.

The condition (i) implies that $f'(\cdot)\le 0$. Then, if $\psi(z,\zeta)$ is strictly decreasing in ζ, $f'(x-\psi(z,\zeta))$ is non-positive and decreasing in ζ. Hence, based on Lemma 1, $\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z+\delta,\zeta)\omega(\zeta)d\zeta$ is increasing in δ ($\delta>0$) if $\psi_z(z,\epsilon)$ has the USDP. Similarly, if $\psi(z,\zeta)$ is strictly increasing in ζ, $f'(x-\psi(z,\zeta))$ is non-positive and increasing in ζ. Hence, based on Lemma 1, $\int_{\underline{\epsilon}}^{\bar{\epsilon}} f'(x-\psi(z,\zeta))\psi_z(z+\delta,\zeta)\omega(\zeta)d\zeta$ is increasing in δ ($\delta>0$) if $\psi_z(z,\epsilon)$ has the LSDP. We thus have shown that, under the conditions (i) and (ii), the inequalities (18) and (19) hold and thus the joint concavity of $F(x,z)$ is guaranteed. ■

Based on Assumptions 2 and 3, we derive the following result with Leibniz integral rule (Flanders, 1973).

LEMMA A.1. *If $V_{t+1}(\cdot)$ is continuously differentiable, then $U_t(p_t,y_t)$ is twice continuously differentiable.*

Proof of Lemma A.1.

Note that

$$
\begin{aligned}
\mathcal{C}(y_t-d_t(p_t,\zeta)) =& h[y_t-d_t(p_t,\zeta)]^+ + b[d_t(p_t,\zeta)-y_t]^+ \\
=&(h+b)[y_t-d_t(p_t,\zeta)]^+ - b(y_t-d_t(p_t,\zeta)).
\end{aligned}
$$

According to the Leibniz integral rule, the first order derivatives of $U_t(p_t,y_t)$ are denoted by

$$
\begin{aligned}
\frac{\partial U_t(p_t,y_t)}{\partial p_t} =& \int_{\underline{\epsilon}}^{\bar{\epsilon}} \left[d_t(p_t,\zeta)+(p_t+\beta M-b)d_{t,p}(p_t,\zeta)-\beta V'_{t+1}(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)\right] w_t(\zeta)d\zeta \\
&+(h+b)\int_{\bar{\zeta}_t}^{\bar{\epsilon}} d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta, \\
\frac{\partial U_t(p_t,y_t)}{\partial y_t} =& -\beta M + b - (h+b)\int_{\bar{\zeta}_t}^{\bar{\epsilon}} w_t(\zeta)d\zeta + \beta\int_{\underline{\epsilon}}^{\bar{\epsilon}} V'_{t+1}(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta,
\end{aligned}
$$

i.e., the differentiation operator and the integration operator can be interchanged in the above formulae. The first order derivatives are continuous since $V'_{t+1}(\cdot)$, $d_t(p_t,\zeta)$, and $d_{t,p}(p_t,\zeta)$ are continuous. Through the integration by parts,

$$
\begin{aligned}
\frac{\partial U_t(p_t,y_t)}{\partial y_t} =& -\beta M + b - (h+b)\int_{\bar{\zeta}_t}^{\bar{\epsilon}} w_t(\zeta)d\zeta - \beta\left[\frac{w_t(\bar{\epsilon})}{d_{t,\zeta}(p_t,\bar{\epsilon})}V_{t+1}(y_t-d_t(p_t,\bar{\epsilon}))\right] \\
&+\beta\left[\frac{w_t(\underline{\epsilon})}{d_{t,\zeta}(p_t,\underline{\epsilon})}V_{t+1}(y_t-d_t(p_t,\underline{\epsilon}))\right] + \beta\int_{\underline{\epsilon}}^{\bar{\epsilon}} V_{t+1}(y_t-d_t(p_t,\zeta))d\left(\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}\right).
\end{aligned}
$$

We then use the first order derivatives to show that all the second order partial derivatives exist and are continuous. For the sake of brevity, we prove the claim for $\frac{\partial^2 U_t(p_t,y_t)}{\partial y_t \partial p_t}$. The proof of the remaining second order partial derivatives $\frac{\partial^2 U_t(p_t,y_t)}{\partial y_t^2}$ and $\frac{\partial^2 U_t(p_t,y_t)}{\partial p_t^2}$ is analogous.

Since $d_t(p_t, \zeta)$ is strictly decreasing in ζ as in Assumption 3, $d_{t,\zeta}(p_t, \zeta) \neq 0$ for any ζ in the support of ϵ_t and $p_t \in [\underline{p}, \overline{p}]$. Then $\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}$ must be a limited value. With the Leibniz integral rule, the interchange of differentiation and integration operators is valid and hence the second order partial derivative is denoted by

$$\begin{aligned}\frac{\partial^2 U_t(p_t,y_t)}{\partial y_t \partial p_t} =&(h+b)w_t(\bar{\zeta}_t)\frac{\partial \bar{\zeta}_t}{\partial p_t} + \beta \frac{w_t(\bar{\epsilon})d_{t,p\zeta}(p_t,\bar{\epsilon})}{d^2_{t,\zeta}(p_t,\bar{\epsilon})}V_{t+1}(y_t - d_t(p_t,\bar{\epsilon})) + \beta \frac{w_t(\bar{\epsilon})d_{t,p}(p_t,\bar{\epsilon})}{d_{t,\zeta}(p_t,\bar{\epsilon})}V'_{t+1}(y_t - d_t(p_t,\bar{\epsilon}))\\ &- \beta \frac{w_t(\underline{\epsilon})d_{t,p\zeta}(p_t,\underline{\epsilon})}{d^2_{t,\zeta}(p_t,\underline{\epsilon})}V_{t+1}(y_t - d_t(p_t,\underline{\epsilon})) - \beta \frac{w_t(\underline{\epsilon})d_{t,p}(p_t,\underline{\epsilon})}{d_{t,\zeta}(p_t,\underline{\epsilon})}V'_{t+1}(y_t - d_t(p_t,\underline{\epsilon}))\\ &- \beta \int_{\underline{\epsilon}}^{\bar{\epsilon}} V'_{t+1}(y_t - d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)\frac{w'_t(\zeta)d_{t,\zeta}(p_t,\zeta) - w_t(\zeta)d_{t,\zeta\zeta}(p_t,\zeta)}{d^2_{t,\zeta}(p_t,\zeta)}d\zeta\\ &+ \beta \int_{\underline{\epsilon}}^{\bar{\epsilon}} V_{t+1}(y_t - d_t(p_t,\zeta))\frac{w'_t(\zeta)d_{t,p\zeta}(p_t,\zeta) - w_t(\zeta)d_{t,p\zeta\zeta}(p_t,\zeta)}{d^2_{t,\zeta}(p_t,\zeta)}d\zeta\\ &- 2\beta \int_{\underline{\epsilon}}^{\bar{\epsilon}} V_{t+1}(y_t - d_t(p_t,\zeta))\frac{[w'_t(\zeta)d_{t,\zeta}(p_t,\zeta) - w_t(\zeta)d_{t,\zeta\zeta}(p_t,\zeta)]\, d_{t,p\zeta}(p_t,\zeta)}{d^3_{t,\zeta}(p_t,\zeta)}d\zeta.\end{aligned}$$

The above second order partial derivative exists and is continuous since $d_t^{-1}(p_t, y_t)$ and $w_t(\zeta)$ are once continuously differentiable, and $d_t(p_t, \zeta)$ is thrice continuously differentiable. We thus have shown that $U_t(p_t, y_t)$ is twice continuously differentiable when $V_{t+1}(\cdot)$ is continuously differentiable in the backorder model. ■

Proof of Lemma 2.

(1) If $V_{t+1}(\cdot)$ is concave, then $G_t(\cdot)$ in (6) must be concave since $G_t(z) = -\mathcal{C}(z) + \beta V_{t+1}(z)$ and $\mathcal{C}(\cdot)$ is convex. As shown in Lemma A.1, $U_t(p_t, y_t)$ is twice continuously differentiable if $V_{t+1}(\cdot)$ is continuously differentiable. Note that

$$\frac{\partial U_t(p_t,y_t)}{\partial y_t} = -\beta M + \int_{\underline{\epsilon}}^{\bar{\epsilon}} G'_t(y_t - d_t(p_t,\zeta))w_t(\zeta)d\zeta.$$

Since $G_t(y_t - d_t(p_t, \zeta))$ is concave, $G'_t(y_t - d_t(p_t, \zeta))$ must be increasing in $d_t(p_t, \zeta)$, and hence decreasing in p_t (since $d_t(p_t, \zeta)$ is decreasing in p_t). As a result, we must have

$$\frac{\partial^2 U_t(p_t,y_t)}{\partial y_t \partial p_t} \leq 0.$$

Hence, $U_t(p_t, y_t)$ is submodular in (y_t, p_t) when $V_{t+1}(\cdot)$ is concave.

Next, we show that $p^*_t(y_t)$ is decreasing in y_t when $U_t(p_t, y_t)$ is submodular in (p_t, y_t) in period t. Note that, for $U_t(p_t, y_t)$, being submodular in (p_t, y_t) is equivalent to being supermodular in $(-p_t, y_t)$. Hence, based on Theorem 2.7.5 of Topkis (1998), we directly have that the optimal price w.r.t the order-up-to level, denoted by $p^*_t(y_t)$, is decreasing in y_t.

(2) Let $y^*_{t+1}(x_{t+1}) = \arg\max_{y_{t+1} \geq x_{t+1}} U_{t+1}(p^*_{t+1}(y_{t+1}), y_{t+1})$. Based on the envelope theorem (Milgrom and Segal 2002) and the equation (4), if $y^*_{t+1}(x_{t+1}) > x_{t+1}$, we must have that $V'_{t+1}(x_{t+1}) = M$. Note that, as $x_{t+1} \to -\infty$, it must be optimal to order a positive quantity of products and hence $y^*_{t+1}(x_{t+1}) > x_{t+1}$ holds in this case. Then, $V'_{t+1}(x_{t+1}) = M$ must hold when $x_{t+1} \to -\infty$. Since we assume $V_{t+1}(\cdot)$ is concave, the first order derivative $V'_{t+1}(x_{t+1})$ is decreasing in x_{t+1}. Hence, for all x_{t+1}, we have that $V'_{t+1}(x_{t+1}) \leq M$.

Since $V_{t+1}(\cdot)$ is concave, $G_t(x_{t+1})$ is concave as well. Note that,

$$G'_t(x_{t+1}) = -\mathcal{C}'(x_{t+1}) + \beta V'_{t+1}(x_{t+1})$$

$$= \begin{cases} -h + \beta V'_{t+1}(x_{t+1}) & x_{t+1} \geq 0, \\ b + \beta V'_{t+1}(x_{t+1}) & x_{t+1} < 0. \end{cases}$$

Recall that $V'_{t+1}(x_{t+1}) \leq M$ for any x_{t+1}. Hence, $G'_t(x_{t+1}) \leq 0$ when $M \leq -b/\beta$. ∎

We next show that, in the backorder model, the joint concavity of $U_t(p_t, y_t)$ would lead to the continuous differentiability of value functions.

Lemma A.2. *If $U_t(p_t, y_t)$ is jointly concave in (p_t, y_t), then $V_t(x_t)$ is continuously differentiable.*

Proof of Lemma A.2.

This result is shown by an inductive proof and the logic is as follows. We know that $V_{T+1}(\cdot)$ is continuously differentiable. Suppose that $V_{t+1}(\cdot)$ is continuously differentiable, Lemma A.1 has shown that $U_t(p_t, y_t)$ is twice continuously differentiable. Note that, if $U_t(p_t, y_t)$ is jointly concave in (p_t, y_t), then it is strictly concave since the demand functions are continuous (see the proof of Theorem 1 in Federgruen and Yang 2011). Based on the twice continuous differentiability and the strict concavity of $U_t(p_t, y_t)$, we shall show that $V_t(\cdot)$ is continuously differentiable.

Recall that $V_t(x_t) = \max_{\underline{p} \leq p_t \leq \overline{p}, y_t \geq x_t} [U_t(p_t, y_t) + Mx_t]$. If $y_t > x_t$, based on the envelope theorem, we have $V'_t(x_t) = M$ for $t = 1, \cdots, T$, which is continuously differentiable. If $y_t = x_t$, then we shall show the continuous differentiability of $V_t(x_t)$ following Federgruen and Yang (2011). Note that $U_t(p_t, y_t)$ is strictly concave, and the constraint $\underline{p} \leq p_t \leq \overline{p}$ is linear. According to the Theorem 7.3 in Fiacco and Kyparisis (1985), we conclude that $V_t(x_t)$ is continuously differentiable if $y_t = x_t$. ∎

Proof of Theorem 1.

We show the results by an inductive proof. At the period $T+1$, it is clear that $V_{T+1}(x_{T+1}) = (M+\alpha)x_{T+1}$ is concave and continuously differentiable. Suppose that $V_{t+1}(x_{t+1})$ is concave and continuously differentiable. Then, it is sufficient to show that $V_t(x_t)$ has the same properties as well. Recall that $G_t(y_t - d_t(p_t, \zeta)) = -\mathcal{C}(y_t - d_t(p_t, \zeta)) + \beta V_{t+1}(y_t - d_t(p_t, \zeta))$ and $R_t(p_t, \zeta) = (p_t + \beta M) d_t(p_t, \zeta)$. Then,

$$\begin{aligned} V_t(x_t) &= \max_{\underline{p} \leq p_t \leq \overline{p},\, y_t \geq x_t} [U_t(p_t, y_t) + Mx_t] \\ &= \max_{\underline{p} \leq p_t \leq \overline{p},\, y_t \geq x_t} \int_{\underline{\epsilon}}^{\overline{\epsilon}} [R_t(p_t, \zeta) - \beta M y_t + Mx_t + G_t(y_t - d_t(p_t, \zeta))] w_t(\zeta) d\zeta. \end{aligned}$$

We first show the joint concavity of $U_t(p_t, y_t)$. Based on the above equation, the joint concavity of $U_t(p_t, y_t)$ can be guaranteed if (1) $\int_{\underline{\epsilon}}^{\overline{\epsilon}} G_t(y_t - d_t(p_t, \zeta)) w_t(\zeta) d\zeta$ is jointly concave in (p_t, y_t) and (2) $\int_{\underline{\epsilon}}^{\overline{\epsilon}} R_t(p_t, \zeta) w_t(\zeta) d\zeta$ is concave in p_t. In the following, we sequentially show that, under the condition in Theorem 1, the preceding requirements (1) and (2) are satisfied.

Since $V_{t+1}(\cdot)$ is continuously differentiable, based on Lemma A.2, $U_t(p_t, y_t)$ is twice continuously differentiable. Therefore, $G_t(\cdot)$ must be twice continuously differentiable. Let $\Omega_t(p_t, y_t) = \int_{\underline{\epsilon}}^{\overline{\epsilon}} G_t(y_t - d_t(p_t, \zeta)) w_t(\zeta) d\zeta$. Note that the form of $\Omega_t(p_t, y_t)$ is consistent with the form of $F(x, z)$ in Proposition 1. In addition, the requirements for the continuous differentiability of $G_t(\cdot)$ and $d_t(p_t, \zeta)$ are satisfied. Hence, based on Proposition 1, we conclude that $\Omega_t(p_t, y_t)$ is jointly concave in (p_t, y_t) under the conditions

(a) $G_t(y_t - d_t(p_t,\zeta))$ is a nonincreasing and concave function;

(b) $d_t(p_t,\zeta)$ is strictly decreasing in ζ and $d_{t,p}(p_t,\zeta)$ has the USDP.

Since $V_{t+1}(\cdot)$ is concave, $G_t(y_t - d_t(p_t,\zeta))$ is concave as well. Recall that in Lemma 2, $G'_t(y_t - d_t(p_t,\zeta)) \le 0$ for $M = -b/\beta$. Hence, the condition (a) must be satisfied. The condition (b) is exactly the condition in Theorem 1. Hence, we have shown that the requirement (1) is satisfied under the condition in Theorem 1.

For the requirement (2), when $M = -b/\beta$, $\int_{\underline{\epsilon}}^{\overline{\epsilon}} R_t(p_t,\zeta)w_t(\zeta)d\zeta$ is concave in p_t if and only if

$$2\int_{\underline{\epsilon}}^{\overline{\epsilon}} d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta + (p_t - b)\int_{\underline{\epsilon}}^{\overline{\epsilon}} d_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta \le 0.$$

Note that $\int_{\underline{\epsilon}}^{\overline{\epsilon}} d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta \le 0$ must be valid due to Assumption 1; $p_t \ge b$ holds in the backorder model as we discussed before; the condition in Theorem 1 implies that $\int_{\underline{\epsilon}}^{\overline{\epsilon}} d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta$ is decreasing in p_t, i.e., $\int_{\underline{\epsilon}}^{\overline{\epsilon}} d_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta \le 0$. Therefore, the requirement (2) is satisfied under the condition in Theorem 1. We thus have shown that, under the condition in Theorem 1, the requirements (1) and (2) are both satisfied and hence $U_t(p_t,y_t)$ must be jointly concave.

Since the demand functions are continuous, $U_t(p_t,y_t)$ is strictly concave and hence there exist a unique $S_t = \arg\max_{y_t} U_t(p_t^*(y_t),y_t)$ and a unique $p_t^*(y_t) = \arg\max_{\underline{p}\le p_t\le\overline{p}} U_t(p_t,y_t)$ given y_t. The joint concavity of $U_t(p_t,y_t)$ implies that $V_t(x_t)$ is continuously differentiable based on Lemma A.2. Moreover, $V_t(x_t)$ is concave in x_t due to the following reason. Given the initial inventory level x_t, if the optimal order-up-to level $y^*(x_t) > x_t$, then $V'_t(x_t) = M$ based on the envelope theorem and hence $V_t(x_t)$ must be concave; if $y^*(x_t) = x_t$, then $V_t(x_t)$ is concave as $U_t(p_t,y_t)$ is jointly concave (see page 88 of Chapter 3.2.5, Boyd and Vandenberghe, 2004).

In addition, based on Lemma 2, the concavity of $V_{t+1}(x_t)$ implies the submodularity of $U_t(p_t,y_t)$ in the backorder model. Since $U_t(p_t,y_t)$ is submodular and jointly concave in (p_t,y_t), it follows that a BSLP policy must be optimal: if $x_t < S_t$, it is optimal to order the inventory level up to S_t and set price $p_t^*(S_t)$; otherwise, it is optimal not to order and set price $p_t^*(x_t)$. ■

For the lost-sales model, we first show the following result based on Assumptions 2 and 3.

LEMMA A.3. *If $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable, then $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable.*

Proof of Lemma A.3.

According to the Leibniz integral rule, the first order derivatives of $\widehat{U}_t(p_t,y_t)$ are denoted by

$$\begin{aligned}\frac{\partial \widehat{U}_t(p_t,y_t)}{\partial p_t} =& y_t - \int_{\bar{\zeta}_t}^{\overline{\epsilon}} \Big[(y_t - d_t(p_t,\zeta)) - (p_t+h)d_{t,p}(p_t,\zeta) + \beta\widehat{V}'_{t+1}(y_t - d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)\Big] w_t(\zeta)d\zeta,\\ \frac{\partial \widehat{U}_t(p_t,y_t)}{\partial y_t} =& p_t + \int_{\bar{\zeta}_t}^{\overline{\epsilon}} \Big[-(p_t+h) + \beta\widehat{V}'_{t+1}(y_t - d_t(p_t,\zeta))\Big] w_t(\zeta)d\zeta,\end{aligned}$$

i.e., the differentiation operator and the integration operator can be interchanged in the above formulas. The first order derivatives are continuous since $\widehat{V}'_{t+1}(\cdot)$, $d_t(p_t,\zeta)$, and $d_{t,p}(p_t,\zeta)$ are continuous. We then use the first order derivatives to show that all the second order partial derivatives exist and are continuous. For the sake of brevity, we prove the claim for $\frac{\partial^2 \widehat{U}_t(p_t,y_t)}{\partial y_t \partial p_t}$. The proof for the remaining second order partial derivatives $\frac{\partial^2 \widehat{U}_t(p_t,y_t)}{\partial y_t^2}$ and $\frac{\partial^2 \widehat{U}_t(p_t,y_t)}{\partial p_t^2}$ is analogous. Based on the integration by parts, we can rewrite $\frac{\partial \widehat{U}_t(p_t,y_t)}{\partial y_t}$ as

$$\frac{\partial \widehat{U}_t(p_t,y_t)}{\partial y_t} = p_t - (p_t+h)\int_{\bar{\zeta}_t}^{\overline{\epsilon}} w_t(\zeta)d\zeta - \beta\left[\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}\widehat{V}_{t+1}(y_t - d_t(p_t,\zeta))\right]_{\bar{\zeta}_t}^{\overline{\epsilon}}$$

$$
\begin{aligned}
&+\beta\int_{\bar{\zeta}_t}^{\bar{\epsilon}}\widehat{V}_{t+1}(y_t-d_t(p_t,\zeta))d\left(\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}\right)\\
=&p_t-(p_t+h)\int_{\bar{\zeta}_t}^{\bar{\epsilon}}w_t(\zeta)d\zeta-\beta\left[\frac{w_t(\bar{\epsilon})}{d_{t,\zeta}(p_t,\bar{\epsilon})}\widehat{V}_{t+1}(y_t-d_t(p_t,\bar{\epsilon}))\right]\\
&+\beta\left[\frac{w_t(\bar{\zeta}_t)}{d_{t,\zeta}(p_t,\bar{\zeta}_t)}\widehat{V}_{t+1}(y_t-d_t(p_t,\bar{\zeta}_t))\right]+\beta\int_{\bar{\zeta}_t}^{\bar{\epsilon}}\widehat{V}_{t+1}(y_t-d_t(p_t,\zeta))d\left(\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}\right).
\end{aligned}
$$

Since $d_t(p_t,\zeta)$ is strictly decreasing in ζ as in Assumption 3, $d_{t,\zeta}(p_t,\zeta)\neq 0$ for any ζ in the support of ϵ_t and $p_t\in[\underline{p},\overline{p}]$. Then $\frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}$ must be a limited value. With the Leibniz integral rule, the interchange of differentiation and integration operators is valid and hence the second order partial derivative is denoted by

$$
\begin{aligned}
\frac{\partial^2\widehat{U}_t(p_t,y_t)}{\partial y_t\partial p_t}=&W_t(\bar{\zeta}_t)+(p_t+h)w_t(\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial p_t}\\
&+\beta\frac{w_t(\bar{\epsilon})d_{t,p\zeta}(p_t,\bar{\epsilon})}{d^2_{t,\zeta}(p_t,\bar{\epsilon})}\widehat{V}_{t+1}(y_t-d_t(p_t,\bar{\epsilon}))+\beta\frac{w_t(\bar{\epsilon})d_{t,p}(p_t,\bar{\epsilon})}{d_{t,\zeta}(p_t,\bar{\epsilon})}\widehat{V}'_{t+1}(y_t-d_t(p_t,\bar{\epsilon}))\\
&-\beta\frac{w_t(\bar{\zeta}_t)}{d_{t,\zeta}(p_t,\bar{\zeta}_t)}\widehat{V}'_{t+1}(y_t-d_t(p_t,\bar{\zeta}_t))\left(d_{t,\zeta}(p_t,\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial p_t}+d_{t,p}(p_t,\bar{\zeta}_t)\right)\\
&+\beta\widehat{V}_{t+1}(y_t-d_t(p_t,\bar{\zeta}_t))\left(\frac{w'_t(\bar{\zeta}_t)d_{t,\zeta}(p_t,\bar{\zeta}_t)-w_t(\bar{\zeta}_t)d_{t,\zeta\zeta}(p_t,\bar{\zeta}_t)}{d^2_{t,\zeta}(p_t,\bar{\zeta}_t)}\frac{\partial\bar{\zeta}_t}{\partial p_t}-\frac{w_t(\bar{\zeta}_t)d_{t,p\zeta}(p_t,\bar{\zeta}_t)}{d^2_{t,\zeta}(p_t,\bar{\zeta}_t)}\right)\\
&-\beta\int_{\bar{\zeta}_t}^{\bar{\epsilon}}\widehat{V}'_{t+1}(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)\frac{w'_t(\zeta)d_{t,\zeta}(p_t,\zeta)-w_t(\zeta)d_{t,\zeta\zeta}(p_t,\zeta)}{d^2_{t,\zeta}(p_t,\zeta)}d\zeta\\
&+\beta\int_{\bar{\zeta}_t}^{\bar{\epsilon}}\widehat{V}_{t+1}(y_t-d_t(p_t,\zeta))\frac{w'_t(\zeta)d_{t,p\zeta}(p_t,\zeta)-w_t(\zeta)d_{t,p\zeta\zeta}(p_t,\zeta)}{d^2_{t,\zeta}(p_t,\zeta)}d\zeta\\
&-2\beta\int_{\bar{\zeta}_t}^{\bar{\epsilon}}\widehat{V}_{t+1}(y_t-d_t(p_t,\zeta))\frac{\left[w'_t(\zeta)d_{t,\zeta}(p_t,\zeta)-w_t(\zeta)d_{t,\zeta\zeta}(p_t,\zeta)\right]d_{t,p\zeta}(p_t,\zeta)}{d^3_{t,\zeta}(p_t,\zeta)}d\zeta\\
&-\beta\widehat{V}_{t+1}(y_t-d_t(p_t,\bar{\zeta}_t))\frac{w'_t(\bar{\zeta}_t)d_{t,\zeta}(p_t,\bar{\zeta}_t)-w_t(\bar{\zeta}_t)d_{t,\zeta\zeta}(p_t,\bar{\zeta}_t)}{d^2_{t,\zeta}(p_t,\bar{\zeta}_t)}\frac{\partial\bar{\zeta}_t}{\partial p_t}.
\end{aligned}
$$

The above second order partial derivative exits and is continuous since $d_t^{-1}(p_t,y_t)$ and $w_t(\zeta)$ are once continuously differentiable, and $d_t(p_t,\zeta)$ is thrice continuously differentiable. We thus have shown that $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable when $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable in the lost-sales model. ■

With Lemma A.3, we then show the sufficient condition for the submodularity of $\widehat{U}_t(p_t,y_t)$ and also a functional property of $G_t([y_t-d_t(p_t,\epsilon_t)]^+)$ in (10) that is critical for our analysis.

LEMMA A.4. *Suppose that $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable and concave, and the single-period expected profit function $Q_t(p_t,y_t)$ is submodular in (p_t,y_t), then we have the following two results:*

(1) $\widehat{V}_{t+1}(x_{t+1})$ is deceasing in x_{t+1} and $\widehat{V}'_{t+1}(0)=0$ for $t=0,\cdots,T-1$, also $G'_t(0)=-h$ for $G_t(\cdot)$ in (10) and $t=1,\cdots,T$;

(2) $\widehat{U}_t(p_t,y_t)$ is submodular in (p_t,y_t) and as a result the smallest optimal price given order-up-to level, denoted by $p^*_t(y_t)=\min\arg\max_{\underline{p}\le p_t\le\overline{p}}\widehat{U}_t(p_t,y_t)$, is decreasing in y_t.

Proof of Lemma A.4.

We first prove that $\widehat{V}_{t+1}(x_{t+1})$ is decreasing in x_{t+1}. This follows immediately from the optimization problem (8): as x_{t+1} increases, the feasible set $[\underline{p},\overline{p}]\times[x_{t+1},\infty)$ shrinks so that the maximum value $\widehat{V}_{t+1}(x_{t+1})$ decreases. Then, in the following, we shall show that $\widehat{V}'_{t+1}(0)=0$, $G'_t(0)=-h$, $\widehat{U}_t(p_t,y_t)$ is submodular in (p_t,y_t), and $p^*_t(y_t)$ is decreasing in y_t when $\widehat{U}_t(p_t,y_t)$ is submodular.

As $\widehat{V}_{t+1}(\cdot)$ is concave and continuously differentiable, $G_t([y_t - d_t(p_t,\epsilon_t)]^+)$ is concave, and $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable due to Lemma A.3. Note that

$$\frac{\partial^2 \widehat{U}_t(p_t,y_t)}{\partial y_t \partial p_t} = W_t(\bar{\zeta}_t) - \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) d_{t,p}(p_t,\zeta) w_t(\zeta) d\zeta + (p_t - G_t'(0)) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial p_t},$$

where $G_t''(y_t - d_t(p_t,\zeta)) \le 0$ for any $\zeta \ge \bar{\zeta}_t$ due to the concavity of $G_t([y_t - d_t(p_t,\epsilon_t)]^+)$ and Assumption 3, and $d_{t,p}(p_t,\zeta) \le 0$ for any ζ due to Assumption 1. Then, $\widehat{U}_t(p_t,y_t)$ is submodular in (y_t,p_t) under the sufficient condition

$$W_t(\bar{\zeta}_t) + (p_t - G_t'(0)) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial p_t} \le 0.$$

Note that, when $Q_t(p_t,y_t)$ is submodular in (p_t,y_t),

$$\frac{\partial^2 Q_t(p_t,y_t)}{\partial y_t \partial p_t} = W_t(\bar{\zeta}_t) + (p_t + h) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial p_t} \le 0.$$

Hence, as long as $G_t'(0) \le -h$, the submodularity of $\widehat{U}_t(p_t,y_t)$ holds if $Q_t(p_t,y_t)$ is submodular in (p_t,y_t). We then show that in fact $\widehat{V}_{t+1}'(0) = 0$ and $G_t'(0) = -h$ in the following.

Let $\epsilon_t(\omega)$ be a realization of ϵ_t and $x_{t+1}^{\omega} = [y_t - d_t(p_t,\epsilon_t(\omega))]^+$ denote the corresponding inventory state in period $t+1$. Then, $G_t(x_{t+1}^{\omega}) = -h x_{t+1}^{\omega} + \beta \widehat{V}_{t+1}(x_{t+1}^{\omega})$ and $G_t'(x_{t+1}^{\omega}) = -h + \beta \widehat{V}_{t+1}'(x_{t+1}^{\omega})$ for any $x_{t+1}^{\omega} \ge 0$. Let $y_{t+1}^*(x_{t+1}^{\omega})$ be the optimal solution to $\max_{y_{t+1} \ge x_{t+1}^{\omega}} \widehat{U}_{t+1}(p^*(y_{t+1}), y_{t+1})$ in period $t+1$ when given the initial inventory level x_{t+1}^{ω}. Then, if $y_{t+1}^*(0) > 0$, based on the envelope theorem, we have that $\widehat{V}_{t+1}'(0) = 0$ and $G_t'(0) = -h$. If $y_{t+1}^*(0) = 0$, i.e., it is optimal to order nothing and keep the inventory level to be 0, then $\widehat{V}_{t+1}(x_{t+1}^{\omega}) = \int_{-\infty}^{+\infty} \beta \widehat{V}_{t+2}(0) w_t(\zeta) d\zeta$ for $x_{t+1}^{\omega} = 0$. Again, $\widehat{V}_{t+1}'(0) = 0$ holds and hence $G_t'(0) = -h$. Up to now, we have shown that $\widehat{V}_{t+1}'(0) = 0$ and $G_t'(0) = -h$ holds.

As $G_t'(0) = -h$, $\widehat{U}_t(p_t,y_t)$ must be submodular in (p_t,y_t) if $\widehat{V}_{t+1}(\cdot)$ is concave and $Q_t(p_t,y_t)$ is submodular in (p_t,y_t) based on our previous analysis. According to the same argument shown in the proof of Lemma 2, the decreasing property of $p_t^*(y_t)$ w.r.t y_t holds for the submodular function $\widehat{U}_t(p_t,y_t)$. ■

Proof of Lemma 3.

For the ease of exposition, we define

$$H_t(p_t,y_t) = \int_{\underline{\epsilon}}^{\bar{\epsilon}} \left[-p_t[y_t - d_t(p_t,\zeta)]^+ + G_t([y_t - d_t(p_t,\zeta)]^+) \right] w_t(\zeta) d\zeta,$$

where $G_t([y_t - d_t(p_t,\zeta)]^+)$ has been defined in (10). Then, $\widehat{U}_t(p_t,y_t) = p_t y_t + H_t(p_t,y_t)$.

Since $\widehat{V}_t(\cdot)$ is continuously differentiable, based on Lemma A.3, $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable. Its first order derivatives and second order partial derivatives are

$$\begin{cases} \partial \widehat{U}_t(p_t,y_t)/\partial p_t = y_t + \partial H_t(p_t,y_t)/\partial p_t, \\ \partial \widehat{U}_t(p_t,y_t)/\partial y_t = p_t + \partial H_t(p_t,y_t)/\partial y_t, \end{cases}$$

where

$$\begin{aligned} \frac{\partial H_t(p_t,y_t)}{\partial y_t} &= \int_{\bar{\zeta}_t}^{\bar{\epsilon}} \left(-p_t + G_t'(y_t - d_t(p_t,\zeta))\right) w_t(\zeta) d\zeta, \\ \frac{\partial H_t(p_t,y_t)}{\partial p_t} &= -\int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t'(y_t - d_t(p_t,\zeta)) d_{t,p}(p_t,\zeta) w_t(\zeta) d\zeta - \int_{\underline{\epsilon}}^{\bar{\epsilon}} [y_t - d_t(p_t,\zeta)]^+ w_t(\zeta) d\zeta \end{aligned}$$

$$+\int_{\bar{\zeta}_t}^{\bar{\epsilon}} p_t d_{t,p}(p_t,\zeta) w_t(\zeta) d\zeta,$$

and

$$\begin{cases} \partial^2 \widehat{U}_t(p_t,y_t)/\partial y_t \partial p_t = 1 + \partial^2 H_t(p_t,y_t)/(\partial y_t \partial p_t), \\ \partial^2 \widehat{U}_t(p_t,y_t)/\partial y_t^2 = \partial^2 H_t(p_t,y_t)/\partial y_t^2, \\ \partial^2 \widehat{U}_t(p_t,y_t)/\partial p_t^2 = \partial^2 H_t(p_t,y_t)/\partial p_t^2, \end{cases}$$

where

$$\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t \partial p_t} = \int_{\bar{\zeta}_t}^{\bar{\epsilon}} \left(-1 - G_t''(y_t - d_t(p_t,\zeta)) d_{t,p}(p_t,\zeta)\right) w_t(\zeta) d\zeta + \left(p_t - G_t'(0)\right) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial p_t}, \tag{23}$$

$$\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t^2} = \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) w_t(\zeta) d\zeta + \left(p_t - G_t'(0)\right) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial y_t}, \tag{24}$$

$$\begin{aligned} \frac{\partial^2 H_t(p_t,y_t)}{\partial p_t^2} = & \int_{\bar{\zeta}_t}^{\bar{\epsilon}} \left[G_t''(y_t - d_t(p_t,\zeta)) d_{t,p}^2(p_t,\zeta) - G_t'(y_t - d_t(p_t,\zeta)) d_{t,pp}(p_t,\zeta)\right] w_t(\zeta) d\zeta \\ & + \int_{\bar{\zeta}_t}^{\bar{\epsilon}} \left(2 d_{t,p}(p_t,\zeta) + p_t d_{t,pp}(p_t,\zeta)\right) w_t(\zeta) d\zeta - \left(p_t - G_t'(0)\right) w_t(\bar{\zeta}_t) d_{t,p}(p_t,\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial p_t}. \end{aligned} \tag{25}$$

To ensure the joint concavity of $\widehat{U}_t(p_t,y_t)$, its Hessian matrix shall be negative semidefinite, i.e.,

$$\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t^2} \leq 0, \tag{26}$$

$$\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t^2} \frac{\partial^2 H_t(p_t,y_t)}{\partial p_t^2} \geq \left(\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t \partial p_t} + 1\right)^2. \tag{27}$$

$$\frac{\partial^2 H_t(p_t,y_t)}{\partial p_t^2} \leq 0, \tag{28}$$

Note that, if both (26) and (27) hold, then the inequality (28) must hold since $\left(\frac{\partial^2 H_t(p_t,y_t)}{\partial y_t \partial p_t} + 1\right)^2 \geq 0$. Hence, we only need to investigate under what conditions the inequalities (26) and (27) hold.

We adopt two facts frequently in the following analysis. First, according to the implicit function theorem (noting that we assume $d_t(p_t,\zeta)$ is strictly monotone in ζ, i.e., $d_{t,\zeta}(p_t,\bar{\zeta}_t) \neq 0$), we have

$$\begin{cases} \partial \bar{\zeta}_t / \partial y_t = 1/d_{t,\zeta}(p_t,\bar{\zeta}_t), \\ \partial \bar{\zeta}_t / \partial p_t = -d_{t,p}(p_t,\bar{\zeta}_t)/d_{t,\zeta}(p_t,\bar{\zeta}_t). \end{cases}$$

Since $d_t(p_t,\zeta)$ is decreasing in p_t and strictly decreasing in ζ as assumed in Assumptions 1 and 3, the inequalities $\frac{\partial \bar{\zeta}_t}{\partial y_t} < 0$ and $\frac{\partial \bar{\zeta}_t}{\partial p_t} \leq 0$ must hold. Second, $G_t'(0) = -h$ holds as discussed in the proof of Lemma A.4 and $G_t''(y_t - d_t(p_t,\zeta)) \leq 0$ due to its concavity.

As $G_t'(0) = -h$ and $\frac{\partial \bar{\zeta}_t}{\partial y_t} < 0$, we have that $(p_t - G_t'(0)) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial y_t} = (p_t + h) w_t(\bar{\zeta}_t) \frac{\partial \bar{\zeta}_t}{\partial y_t} \leq 0$. Further with $G_t''(y_t - d_t(p_t,\zeta)) \leq 0$, it is easy to observe that the inequality (26) must hold through the expression in (24). As a result, we shall focus on the inequality (27) hereafter.

By plugging in the expressions in (23)-(25), the inequality (27) is equivalently expressed by

$$\begin{aligned} & \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) w_t(\zeta) d\zeta \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) d_{t,p}^2(p_t,\zeta) w_t(\zeta) d\zeta \\ & - \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) w_t(\zeta) d\zeta \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t'(y_t - d_t(p_t,\zeta)) d_{t,pp}(p_t,\zeta) w_t(\zeta) d\zeta \\ & + \int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t - d_t(p_t,\zeta)) w_t(\zeta) d\zeta \int_{\bar{\zeta}_t}^{\bar{\epsilon}} \left(2 d_{t,p}(p_t,\zeta) + p_t d_{t,pp}(p_t,\zeta)\right) w_t(\zeta) d\zeta \end{aligned}$$

$$
\begin{aligned}
&-\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)d_{t,p}(p_t,\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\\
&+\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}\int_{\bar\zeta_t}^{\bar\epsilon}\left[G_t''(y_t-d_t(p_t,\zeta))d_{t,p}^2(p_t,\zeta)-G_t'(y_t-d_t(p_t,\zeta))d_{t,pp}(p_t,\zeta)\right]w_t(\zeta)d\zeta\\
&+\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}\int_{\bar\zeta_t}^{\bar\epsilon}(2d_{t,p}(p_t,\zeta)+p_td_{t,pp}(p_t,\zeta))w_t(\zeta)d\zeta\\
&+\left[-\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}\right]\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)d_{t,p}(p_t,\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\right]\\
\geq&W_t^2(\bar\zeta_t)+\left[\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta\right]^2+\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\right]^2\\
&-2W_t(\bar\zeta_t)\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta+2W_t(\bar\zeta_t)\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\\
&-2\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta. \qquad (29)
\end{aligned}
$$

Note that, according to the Cauchy-Schwarz inequality (Steele 2004),

$$
\begin{aligned}
&\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}^2(p_t,\zeta)w_t(\zeta)d\zeta\\
\geq&\left[\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta\right]^2.
\end{aligned}
$$

Since $G_t''(\cdot)\leq 0$ and $-d_{t,p}(p_t,\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}=\frac{\partial\bar\zeta_t}{\partial p_t}$, based on the fundamental inequality ($a^2+b^2\geq 2ab$) and the Cauchy-Schwarz inequality, we also have

$$
\begin{aligned}
&-(p_t-G_t'(0))d_{t,p}(p_t,\bar\zeta_t)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\\
&+(p_t-G_t'(0))w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}^2(p_t,\zeta)w_t(\zeta)d\zeta\\
\geq&-2(p_t-G_t'(0))w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\sqrt{\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}^2(p_t,\zeta)w_t(\zeta)d\zeta}\\
\geq&-2(p_t-G_t'(0))w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta.
\end{aligned}
$$

Moreover, as $-d_{t,p}(p_t,\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}=\frac{\partial\bar\zeta_t}{\partial p_t}$, we have

$$
\left[-\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}\right]\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)d_{t,p}(p_t,\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\right]=\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}\right]^2.
$$

Hence, the inequality (29) holds if

$$
\begin{aligned}
&\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}+\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\right]\int_{\bar\zeta_t}^{\bar\epsilon}(2d_{t,p}(p_t,\zeta)+p_td_{t,pp}(p_t,\zeta))w_t(\zeta)d\zeta\\
&-\left[\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial y_t}+\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\right]\int_{\bar\zeta_t}^{\bar\epsilon}G_t'(y_t-d_t(p_t,\zeta))d_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta\\
\geq&W_t^2(\bar\zeta_t)+2W_t(\bar\zeta_t)\left(p_t-G_t'(0)\right)w_t(\bar\zeta_t)\frac{\partial\bar\zeta_t}{\partial p_t}-2W_t(\bar\zeta_t)\int_{\bar\zeta_t}^{\bar\epsilon}G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta. \qquad (30)
\end{aligned}
$$

We then shall show that the inequality (30) holds under the two conditions in Lemma 3. Since $G_t(\cdot)$ is concave and $G_t'(0)=-h$, we have that $G_t'(y_t-d_t(p_t,\zeta))\leq G_t'(0)\leq 0$ for $\zeta\in[\bar\zeta_t,\bar\epsilon]$. In addition, $\partial\bar\zeta_t/\partial y_t\leq 0$. Under the condition (i), $\int_\mu^{\bar\epsilon}d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta$ is decreasing in p_t for any $\mu\in[\underline{\epsilon},\bar\epsilon]$, which implies that $\int_\mu^{\bar\epsilon}d_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta\leq 0$ for $\mu=\bar\zeta_t$. Then, $\int_{\bar\zeta_t}^{\bar\epsilon}p_td_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta\leq 0$ as $p_t\geq 0$. Since $G_t''(\cdot)\leq 0$, $G_t'(0)=$

$-h<0$, and $d_t(p_t,\zeta)$ is decreasing in ζ, for $\zeta\in[\bar{\zeta}_t,\bar{\epsilon}]$, $G_t'(y_t-d_t(p_t,\zeta))$ is non-positive and decreasing in ζ. Then, based on Lemma 1, $\int_\mu^{\bar{\epsilon}} g(\zeta)d_{t,pp}(p_t,\zeta)w(\zeta)d\zeta\geq 0$ for $\mu=\bar{\zeta}_t$ and $g(\zeta)=G_t'(y_t-d_t(p_t,\zeta))$. Hence, under the condition (i), the following inequalities hold:

$$\begin{cases} \left[(p_t-G_t'(0))\,w_t(\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial y_t}+\int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\right]\int_{\bar{\zeta}_t}^{\bar{\epsilon}}(2d_{t,p}(p_t,\zeta)+p_td_{t,pp}(p_t,\zeta))w_t(\zeta)d\zeta\geq 0,\\ \left[(p_t-G_t'(0))\,w_t(\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial y_t}+\int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t-d_t(p_t,\zeta))w_t(\zeta)d\zeta\right]\int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t'(y_t-d_t(p_t,\zeta))d_{t,pp}(p_t,\zeta)w_t(\zeta)d\zeta\leq 0,\\ -2W_t(\bar{\zeta}_t)\int_{\bar{\zeta}_t}^{\bar{\epsilon}} G_t''(y_t-d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta\leq 0. \end{cases}$$

The condition (ii), i.e.,

$$\frac{\partial^2 Q_t(p_t,y_t)}{\partial y_t\partial p_t}=W_t(\bar{\zeta}_t)+(p_t+h)w_t(\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial p_t}\leq 0,$$

is able to ensure the inequality

$$W_t^2(\bar{\zeta}_t)+2W_t(\bar{\zeta}_t)\,(p_t-G_t'(0))\,w_t(\bar{\zeta}_t)\frac{\partial\bar{\zeta}_t}{\partial p_t}\leq 0,$$

since $G_t'(0)=-h$, $\frac{\partial\bar{\zeta}_t}{\partial p_t}\leq 0$, and $W_t(\cdot)\geq 0$ ($W_t(\cdot)$ is the cumulative distribution function and hence must be non-negative). Therefore, under the conditions (i) and (ii), the inequality (30) must hold and hence $\widehat{U}_t(p_t,y_t)$ must be jointly concave. ■

Once the function $\widehat{U}_t(p_t,y_t)$ is jointly concave, we show that the continuous differentiability of value functions can be guaranteed as follows with the similar argument used for Lemma A.2.

Lemma A.5. *If $\widehat{U}_t(p_t,y_t)$ is jointly concave in (p_t,y_t), then $\widehat{V}_t(x_t)$ is continuously differentiable.*

Proof of Lemma A.5.

We show the results by an inductive proof and the logic is as follows. We know that $\widehat{V}_{T+1}(\cdot)$ is continuously differentiable. Suppose that $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable, Lemma A.3 has shown that $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable. Note that, if $\widehat{U}_t(p_t,y_t)$ is jointly concave in (p_t,y_t), then it is strictly concave since the demand distributions are continuous (see the proof of Theorem 1 in Federgruen and Yang 2011). Based on the twice continuous differentiability and the strict concavity of $\widehat{U}_t(p_t,y_t)$, we shall show that $\widehat{V}_t(\cdot)$ is continuously differentiable.

Recall that $\widehat{V}_t(x_t)=\max_{\underline{p}\leq p_t\leq\bar{p},\,y_t\geq x_t}\widehat{U}_t(p_t,y_t)$. If $y_t>x_t$, based on the envelope theorem (Milgrom and Segal 2002), we have $\widehat{V}_t'(x_t)=0$, which is continuously differentiable. If $y_t=x_t$, then we shall show the continuous differentiability of $\widehat{V}_t(x_t)$ following Federgruen and Yang (2011). Note that $\widehat{U}_t(p_t,y_t)$ is strictly concave, and the constraint $\underline{p}\leq p_t\leq\bar{p}$ is linear. Hence, according to the Theorem 7.3 in Fiacco and Kyparisis (1985), we can conclude that $\widehat{V}_t(x_t)$ is continuously differentiable if $y_t=x_t$. ■

Proof of Theorem 2.

We show the results by an inductive proof. It is clear that $\widehat{V}_{T+1}(\cdot)$ is continuously differentiable and concave. Assume that $\widehat{V}_{t+1}(\cdot)$ is continuously differentiable and concave, then $\widehat{U}_t(p_t,y_t)$ is twice continuously differentiable as shown in Lemma A.3. It is sufficient to show that $\widehat{V}_t(\cdot)$ has the same property.

Based on Lemma 3, the conditions (i) and (ii) in Theorem 2 lead to the joint concavity of $\widehat{U}_t(p_t,y_t)$. In addition, based on Lemma A.4, the condition (ii) leads to the submodularity of $\widehat{U}_t(p_t,y_t)$. Further, based on Lemma A.5, the joint concavity of $\widehat{U}_t(p_t,y_t)$ leads to the fact that $\widehat{V}_t(\cdot)$ must be continuously differentiable.

Note that, given the initial inventory level x_t, if the optimal order-up-to level $y^*(x_t) > x_t$, then $\widehat{V}_t'(x_t) = 0$ based on the envelope theorem and hence $\widehat{V}_t(x_t)$ must be concave; if $y^*(x_t) = x_t$, then $\widehat{V}_t(x_t)$ is concave when $\widehat{U}_t(p_t, y_t)$ is jointly concave. Since $\widehat{U}_t(p_t, y_t)$ is jointly concave, then $\widehat{V}_t(x_t)$ must be concave (see page 88 of Chapter 3.2.5, Boyd and Vandenberghe, 2004).

Up to now, we have shown that $\widehat{V}_t(\cdot)$ is concave and continuously differentiable, and that $\widehat{U}_t(p_t, y_t)$ is twice continuously differentiable, submodular and jointly concave in (p_t, y_t) under the presented conditions. As indicated in the proof of Lemma A.5, when $\widehat{U}_t(p_t, y_t)$ is jointly concave in (p_t, y_t), it must be strictly concave since the demand functions are considered as continuous in our study. Therefore, there are a unique $S_t = \arg\max_{y_t} \widehat{U}_t(p_t^*(y_t), y_t)$ and a unique $p_t^*(y_t) = \arg\max_{\underline{p} \le p_t \le \overline{p}} \widehat{U}_t(p_t, y_t)$ given y_t. Then, by a similar argument as in Theorem 1, a BSLP policy must be optimal due to the submodularity and the joint concavity of $\widehat{U}_t(p_t, y_t)$. ■

Proof of Proposition 2.

The submodularity of the function $Q_t(p_t, y_t)$ in (p_t, y_t) is equivalent to the condition $\partial^2 Q_t(p_t, y_t)/\partial y_t \partial p_t \le 0$. Again we let $\bar{\zeta}_t = d_t^{-1}(p_t, y_t)$, where $d_t^{-1}(p_t, y_t)$ is the inverse function of $d_t(p_t, \epsilon_t)$ w.r.t ϵ_t. Based on Assumption 3, $d_t(p_t, \zeta) \le y_t$ for $\zeta \ge \bar{\zeta}_t$. Then

$$\frac{\partial Q_t(p_t, y_t)}{\partial y_t} = p_t - (p_t + h) \int_{\bar{\zeta}_t}^{\bar{\epsilon}} w_t(\zeta) d\zeta = p_t - (p_t + h)(1 - W_t(\bar{\zeta}_t)),$$

and hence the sufficient and necessary condition $\partial^2 Q_t(p_t, y_t)/(\partial y_t \partial p_t) \le 0$ is equivalent to

$$\frac{\partial^2 Q_t(p_t, y_t)}{\partial y_t \partial p_t} = W_t(\bar{\zeta}_t) + (p_t + h) \frac{\partial W_t(\bar{\zeta}_t)}{\partial p_t} \le 0.$$

Note that $\bar{F}_t(p_t, y_t) = P(d_t(p_t, \epsilon) > y_t) = P(\epsilon_t < \bar{\zeta}_t) = W_t(\bar{\zeta}_t)$ based on Assumption 3. As a result,

$$W_t(\bar{\zeta}_t) + (p_t + h) \frac{\partial W_t(\bar{\zeta}_t)}{\partial p_t} = \bar{F}_t(p_t, y_t) + (p_t + h) \frac{\partial \bar{F}_t(p_t, y_t)}{\partial p_t},$$

i.e., $-(p_t + h) \frac{\partial \bar{F}_t(p_t, y_t)/\partial p_t}{\bar{F}_t(p_t, y_t)} \ge 1$. ■

Proof of Lemma 4.

For the ease of exposition, we define $r_t(\zeta) = \frac{w_t(\zeta)}{\bar{W}_t(\zeta)}$, i.e., $r_t(\zeta_t)$ is the failure rate of ϵ_t. Note that

$$\Delta(p_t, y_t) = -(p_t + h) \frac{\partial}{\partial p_t} \log(\bar{F}_t(p_t, y_t)).$$

Then,

$$\begin{aligned}
\frac{\partial \Delta(p_t, y_t)}{\partial y_t} &= -(p_t + h) \frac{\partial}{\partial p_t} \frac{\partial \bar{F}_t(p_t, y_t)/\partial y_t}{\bar{F}_t(p_t, y_t)}, \\
\frac{\partial \Delta(p_t, y_t)}{\partial p_t} &= -\frac{\partial \bar{F}_t(p_t, y_t)/\partial p_t}{\bar{F}_t(p_t, y_t)} - (p_t + h) \frac{\partial}{\partial p_t} \frac{\partial \bar{F}_t(p_t, y_t)/\partial p_t}{\bar{F}_t(p_t, y_t)}.
\end{aligned}$$

When $d_t(p_t, \epsilon_t)$ is increasing in ϵ_t, we have $\bar{F}_t(p_t, y_t) = \bar{W}_t(\bar{\zeta}_t)$, $\frac{\partial \bar{\zeta}_t}{\partial y_t} = \frac{1}{d_{t,\zeta}(p_t, \bar{\zeta}_t)} \ge 0$, and $\frac{\partial \bar{\zeta}_t}{\partial p_t} = -\frac{d_{t,p}(p_t, \bar{\zeta}_t)}{d_{t,\zeta}(p_t, \bar{\zeta}_t)} \ge 0$. Then,

$$\frac{\partial \Delta(p_t, y_t)}{\partial y_t} = (p_t + h) \frac{\partial}{\partial p_t} \frac{r_t(\bar{\zeta}_t)}{d_{t,\zeta}(p_t, \bar{\zeta}_t)}$$

$$
\begin{aligned}
&=(p_t+h)\left[-r_t'(\bar{\zeta}_t)d_{t,p}(p_t,\bar{\zeta}_t)-r_t(\bar{\zeta}_t)\frac{\partial d_{t,\varsigma}(p_t,\bar{\zeta}_t)}{\partial p_t}\right],\\
\frac{\partial\Delta(p_t,y_t)}{\partial p_t}&=\frac{-r_t(\bar{\zeta}_t)d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}+(p_t+h)\frac{\partial}{\partial p_t}\left[\frac{-r_t(\bar{\zeta}_t)d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\right]\\
&=\frac{-r_t(\bar{\zeta}_t)d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}+(p_t+h)\left\{r_t'(\bar{\zeta}_t)\left(\frac{d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\right)^2-r_t(\bar{\zeta}_t)\frac{\partial}{\partial p_t}\left[\frac{d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\right]\right\}.
\end{aligned}
$$

Then,$\Delta(\ p_t,y_t)$ is increasing in y_t and p_t under the conditions in this lemma.

When $d_t(p_t,\epsilon_t)$ is decreasing in ϵ_t, we have $\bar{F}_t(p_t,y_t)=W_t(\bar{\zeta}_t)$, $\frac{\partial\bar{\zeta}_t}{\partial y_t}=\frac{1}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\le 0$, and $\frac{\partial\bar{\zeta}_t}{\partial p_t}=-\frac{d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\le 0$. Then,

$$
\begin{aligned}
\frac{\partial\Delta(p_t,y_t)}{\partial y_t}&=-(p_t+h)\frac{\partial}{\partial p_t}\left[r_t(\bar{\zeta}_t)\frac{\bar{W}_t(\bar{\zeta}_t)}{W_t(\bar{\zeta}_t)}\frac{1}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\right],\\
\frac{\partial\Delta(p_t,y_t)}{\partial p_t}&=r_t(\bar{\zeta}_t)\frac{\bar{W}_t(\bar{\zeta}_t)}{W_t(\bar{\zeta}_t)}\frac{d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}+(p_t+h)\frac{\partial}{\partial p_t}\left[r_t(\bar{\zeta}_t)\frac{\bar{W}_t(\bar{\zeta}_t)}{W_t(\bar{\zeta}_t)}\frac{d_{t,p}(p_t,\bar{\zeta}_t)}{d_{t,\varsigma}(p_t,\bar{\zeta}_t)}\right].
\end{aligned}
$$

As $\bar{W}_t(\zeta)/W_t(\zeta)$ is decreasing in ζ, under the conditions in this lemma, $\Delta(p_t,y_t)$ is increasing in y_t and p_t.

■

Proof of Corollary 1.

As $S_t\ge \underline{d}_t$, so we only need to verify the submodularity of $Q_t(p_t,y_t)$ for $y_t\ge\underline{d}_t$. When $d_t(p_t,\epsilon_t)$ is increasing in ϵ_t, the submodularity of $Q_t(p_t,y_t)$ holds if

$$
(p_t+h)\frac{w_t(\bar{\zeta}_t)}{\bar{W}_t(\bar{\zeta}_t)}\frac{\partial\bar{\zeta}_t}{\partial p_t}\ge 1.
$$

(1) For $d_t(p_t,\epsilon_t)=\mu_t(p_t)+\epsilon_t$, $\bar{\zeta}_t=y_t-\mu_t(p_t)$ and hence $\frac{\partial\bar{\zeta}_t}{\partial p_t}=-\mu_t'(p_t)$. Then, the submodularity of $Q_t(p_t,y_t)$ holds if

$$
-(p_t+h)\frac{w_t(\bar{\zeta}_t)}{1-W_t(\bar{\zeta}_t)}\mu_t'(p_t)\ge 1.
$$

As $d_t(p_t,\epsilon_t)$ is decreasing in p_t and $\mu_t''(p_t)\le 0$, $d_{t,p}(p_t,\epsilon_t)$ must have the LSDP, also $-\mu_t'(p_t)\ge 0$ and $-\mu_t'(p_t)$ achieves its minimum at $p_t=\underline{p}$. In addition, $\bar{\zeta}_t$ reaches its minimum and accordingly $\frac{w_t(\bar{\zeta}_t)}{1-W_t(\bar{\zeta}_t)}$ reaches its minimum at $(p_t,y_t)=(\underline{p},\underline{d}_t)$ as W_t has the IFR. Hence, $Q_t(p_t,y_t)$ must be submodular for $y_t\ge\underline{d}_t$ as long as

$$
-(\underline{p}+h)w_t(\underline{d}_t-\mu_t(\underline{p}))\mu_t'(\underline{p})\ge 1-W_t(\underline{d}_t-\mu_t(\underline{p})).
$$

(2) For $d_t(p_t,\epsilon_t)=\sigma_t(p_t)\epsilon_t$, $d_{t,p}(p_t,\epsilon_t)$ has the LSDP as $\underline{\epsilon}\ge 0$ and $\sigma_t''\le 0$. Since $\bar{\zeta}_t=\frac{y_t}{\sigma_t(p_t)}$, $\frac{\partial\bar{\zeta}_t}{\partial p_t}=-\frac{y_t\sigma_t'(p_t)}{\sigma_t^2(p_t)}$ and hence the submodularity of $Q_t(p_t,y_t)$ holds if

$$
-(p_t+h)\frac{w_t(\bar{\zeta}_t)}{\bar{W}_t(\bar{\zeta}_t)}\frac{y_t\sigma_t'(p_t)}{\sigma_t^2(p_t)}\ge 1.
$$

Since $d_t(p_t,\epsilon_t)$ is decreasing in p_t, $\sigma_t'(p_t)\le 0$ and hence $\sigma_t(p_t)$ achieves its maximum at $p_t=\underline{p}$. As $-\sigma_t''\ge 0$, $-\sigma_t'$ achieves its minimum at $p_t=\underline{p}$. Then, $-(p_t+h)\frac{\sigma_t'(p_t)}{\sigma^2(p_t)}$ achieves its minimum at $p_t=\underline{p}$. Note that $\bar{\zeta}_t$ achieves its minimum at $(p_t,y_t)=(\underline{p},\underline{d}_t)$. Then, the submodularity of $Q_t(p_t,y_t)$ is guaranteed if $-(\underline{p}+h)w_t\left(\frac{\underline{d}_t}{\sigma_t(\underline{p})}\right)\underline{d}_t\sigma_t'(\underline{p})\ge\bar{W}_t\left(\frac{\underline{d}_t}{\sigma_t(\underline{p})}\right)\sigma_t^2(\underline{p})$.

(3) For $d_t(p_t,\epsilon_t)=\mu_t(p_t)+\sigma_t(p_t)\epsilon_t,\ d_{t,p}(p_t,\epsilon_t)=\mu_t'(p_t)+\sigma_t'(p_t)\epsilon_t \le 0$ and it must have the LSDP as $\underline{\epsilon}\ge 0$ and μ_t,σ_t are decreasing and concave. As $\bar{\zeta}_t=\frac{y_t-\mu_t(p_t)}{\sigma_t(p_t)}$, $\frac{\partial\bar{\zeta}_t}{\partial p_t}=\frac{-\mu_t'(p_t)\sigma_t(p_t)-(y_t-\mu_t(p_t))\sigma_t'(p_t)}{\sigma_t^2(p_t)}$. Then, the submodularity of $Q_t(p_t,y_t)$ holds if

$$(p_t+h)\frac{w_t(\bar{\zeta}_t)}{\bar{W}_t(\bar{\zeta}_t)}\frac{-\mu_t'(p_t)\sigma_t(p_t)-(y_t-\mu_t(p_t))\sigma_t'(p_t)}{\sigma_t^2(p_t)}\ge 1.$$

Note that $\bar{\zeta}_t\ge\frac{\underline{d}_t-\mu_t(\underline{p})}{\sigma_t(\underline{p})}$ and $\frac{-\mu_t'(p_t)\sigma_t(p_t)-(y_t-\mu_t(p_t))\sigma_t'(p_t)}{\sigma_t^2(p_t)}\ge\frac{-\mu_t'(\underline{p})\sigma_t(\overline{p})-(\underline{d}_t-\mu_t(\underline{p}))\sigma_t'(\underline{p})}{\sigma_t^2(\underline{p})}$. Hence, the submodularity of $Q_t(p_t,y_t)$ is guaranteed if $-(\underline{p}+h)w_t\left(\frac{\underline{d}_t-\mu_t(\underline{p})}{\sigma_t(\underline{p})}\right)\left(\mu_t'(\underline{p})\sigma_t(\overline{p})+(\underline{d}_t-\mu_t(\underline{p}))\sigma_t'(\underline{p})\right)\ge\bar{W}_t\left(\frac{\underline{d}_t-\mu_t(\underline{p})}{\sigma_t(\underline{p})}\right)\sigma_t^2(\underline{p})$.

(4) For $d_t(p_t,\epsilon_t)=\ln(bp_t+\epsilon_t)$, $d_{t,p}(p_t,\epsilon_t)=\frac{b}{bp_t+\epsilon_t}\le 0$ as $b\le 0$ and it has the LSDP. As $\bar{\zeta}_t=e^{y_t}-bp_t$, $\frac{\partial\bar{\zeta}_t}{\partial p_t}=-b$ and hence the submodularity of $Q_t(p_t,y_t)$ holds if

$$-b(p_t+h)\frac{w_t(\bar{\zeta}_t)}{\bar{W}_t(\bar{\zeta}_t)}\ge 1.$$

Note that $\bar{\zeta}_t$ is increasing in y_t and p_t. Hence, $\bar{\zeta}_t$ achieves its minimum at $(p_t,y_t)=(\underline{p},\underline{d}_t)$. Then, the submodlarity of $Q_t(p_t,y_t)$ is guaranteed if $-b(\underline{p}+h)w_t(e^{\underline{d}_t}-b\underline{p})\ge\bar{W}_t(e^{\underline{d}_t}-b\underline{p})$. ■

The Backorder Model with Inventory-Dependent Demands

Let $\widehat{V}_t(\cdot)$ the expected optimal discounted cost function from period t and onward. The corresponding dynamic recursion is as follows:

$$\widehat{V}_t(x_t)=\max_{y_t\ge x_t}\mathbb{E}[\widehat{G}_t(y_t-d_t(x_t,\epsilon_t))]+p\mathbb{E}[d_t(x_t,\epsilon_t)],$$

for $t=1,\cdots,T$, where

$$\widehat{G}_t(y_t-d_t(x_t,\zeta))=b(y_t-d_t(x_t,\zeta))-(h+b)[y_t-d_t(x_t,\zeta)]^+ +\beta\widehat{V}_{t+1}(y_t-d_t(x_t,\zeta)).$$

We let $\widehat{V}_{T+1}(x_{T+1})\equiv 0$ for all x_{T+1}. Similar to the basic model, we adopt the following equivalent transformation to reformulate the dynamic recursion. Let $V_t(x_t)=\widehat{V}_t(x_t)+Mx_t$ and then

$$V_t(x_t)=\max_{y_t\ge x_t}\left[\mathbb{E}[G_t(y_t-d_t(x_t,\epsilon_t))]-\beta My_t\right]+(p+\beta M)\mathbb{E}[d_t(x_t,\epsilon_t)]+Mx_t,$$

where

$$G_t(y_t-d_t(x_t,\zeta))=b(y_t-d_t(x_t,\zeta))-(h+b)[y_t-d_t(x_t,\zeta)]^+ +\beta V_{t+1}(y_t-d_t(x_t,\zeta))$$

and $V_{T+1}(x_{T+1})\equiv Mx_{T+1}$.

Proof of Proposition 3.

We show the results by induction. With $M=-b/\beta$, in the terminal period, $V_{T+1}(\cdot)$ is concave and $V_{T+1}'(\cdot)=-b/\beta$. Suppose that $V_{t+1}(\cdot)$ is concave and $V_{t+1}'(\cdot)\le -b/\beta$, it suffices to show that $V_t(\cdot)$ is concave and $V_t'(\cdot)\le -b/\beta$ as well.

As $V_{t+1}(\cdot)$ is concave by the inductive assumption, $G_t(\cdot)$ must be concave as well and

$$G_t'(z_t)=\begin{cases} b+\beta V_{t+1}'(z_t), & z_t\le 0,\\ -h+\beta V_{t+1}'(z_t), & z_t>0,\end{cases}$$

which implies $G_t'(\cdot) \le 0$ since $\beta V_{t+1}'(\cdot) \le -b$. Then, according to Proposition 1, $\mathbb{E}[G_t(y_t - d_t(x_t,\epsilon_t))]$, as well as $\mathbb{E}[G_t(y_t - d_t(x_t,\epsilon_t))] - \beta M y$, is concave in y_t, x_t. Moreover, $(p+\beta M)\mathbb{E}[d_t(x_t,\epsilon_t)]$ is concave in x_t as $d_{t,x}(x_t,\epsilon_t)$ has the USDP/LSDP. Hence, $V_t(x_t)$ must be concave as the feasible region $y_t \ge x_t$ is a convex set.

Let $y_t^*(x_t) = \arg\max_{y_t \ge x_t} [\mathbb{E}[G_t(y_t - d_t(x_t,\epsilon_t))] - \beta M y_t]$. Based on the concavity of $\mathbb{E}[G_t(y_t - d_t(x_t,\epsilon_t))]$ and $(p+\beta M)\mathbb{E}[d_t(x_t,\epsilon_t)]$, we have that, if $x_t < y_t^*(x_t)$, it is optimal to order up to $y_t^*(x_t)$; otherwise, it is optimal to order nothing. As $d_t(x_t,\epsilon_t)$ is decreasing in x_t, $\mathbb{E}[G_t(y_t - d_t(x_t,\epsilon_t))]$ is submodular in (y_t, x_t) as $G_t(\cdot)$ is concave. As a result, $y_t^*(x_t)$ is decreasing in x_t according to Theorem 2.7.5 of Topkis (1998).

We then shall show that $V_t'(x_t) \le -b/\beta$. If $y_t^*(x_t) > x_t$ for $t = 1, \cdots, T$, then based on the envelope theorem and $M = -b/\beta$, we have

$$V_t'(x_t) = \mathbb{E}[G_t'(y_t^*(x_t) - d_t(x_t,\epsilon_t))(-d_{t,x}(x_t,\epsilon_t))] + (p-b)\mathbb{E}[d_{t,x}(x_t,\epsilon_t)] - b/\beta.$$

Since $G_t'(\cdot) \le 0$ and $d_t(x_t,\epsilon_t)$ is decreasing in x_t, we have $V_t'(x_t) \le -b/\beta$. As $y_t^*(x_t) > x_t$ must hold for $x_t \to -\infty$ and $V_t'(x_t)$ is decreasing in x_t due to its concavity, we must have that $V_t'(x_t) \le -b/\beta$ for any x_t. We thus have complete the inductive proof. ■

The Lost-Sales Model with Inventory-Dependent Demands

We formulate the dynamic recursion as follows:

$$\widehat{V}_t(x_t) = \max_{y_t \ge x_t} \left[p y_t + \mathbb{E}[\widehat{H}_t(y_t - d_t(x_t,\epsilon_t))] \right]$$

for $t = 1, \cdots, T$, where

$$\widehat{H}_t(y_t - d_t(x_t,\zeta)) = -(p+h)[y_t - d_t(x_t,\zeta)]^+ + \beta \widehat{V}_{t+1}([y_t - d_t(x_t,\zeta)]^+),$$

and $\widehat{V}_{T+1}(x_{T+1}) \equiv 0$.

Proof of Proposition 4.

Without loss of generality, we assume that $d_t(x_t,\epsilon_t)$ is strictly decreasing in ϵ_t and $d_{t,x}(x_t,\epsilon_t)$ has the USDP. We can obtain the same results when $d_t(x_t,\epsilon_t)$ is strictly increasing in ϵ_t and $d_{t,x}(x_t,\epsilon_t)$ has the LSDP.

We still show the results by induction. Notice that $\widehat{V}_{T+1}(x_{T+1})$ must be decreasing and concave in $x_{T+1} \ge 0$. Suppose that $\widehat{V}_{t+1}(x_{t+1})$ is decreasing and concave in $x_{t+1} \ge 0$, it suffices to show that $\widehat{V}_t(x_t)$ is decreasing and concave in $x_t \ge 0$ as well.

As $\widehat{V}_{t+1}(\cdot)$ is decreasing and concave, $\widehat{H}_t(y_t - d_t(x_t,\zeta))$ must be decreasing and concave in $y_t - d_t(x_t,\zeta)$ and also in y_t. Based on Proposition 1, $\mathbb{E}[\widehat{H}_t(y_t - d_t(x_t,\zeta))]$, as well as $p y_t + \mathbb{E}[\widehat{H}_t(y_t - d_t(x_t,\zeta))]$, is concave in y_t, x_t. Then, $\widehat{V}_t(x_t)$ is concave (concavity is preserved under the maximization operator).

Let $y_t^*(x_t) = \min \arg\max_{y_t \ge x_t} \left[p y_t + \mathbb{E}[\widehat{H}_t(y_t - d_t(x_t,\epsilon_t))] \right]$. If $x_t < y_t^*(x_t)$, then it is optimal to order up to $y_t^*(x_t)$; otherwise, it is optimal to order nothing. As $d_t(x_t,\epsilon_t)$ is decreasing in x_t, then $\mathbb{E}[\widehat{H}_t(y_t - d_t(x_t,\epsilon_t))]$ is submodular in (y_t, x_t) as $\widehat{H}_t(\cdot)$ is concave. Then, $y_t^*(x_t)$ is decreasing in x_t according to Theorem 2.7.5 of Topkis (1998).

We then shall show that $\widehat{V}_t'(x_t) \le 0$ for any $x_t \ge 0$. If $y_t^*(x_t) > x_t$, then

$$\widehat{V}_t'(x_t) = \mathbb{E}[\widehat{H}_t'(y_t - d_t(x_t,\epsilon_t))(-d_{t,x}(x_t,\epsilon_t))].$$

As $\widehat{H}_t(\cdot)$ is decreasing and $d_t(x_t,\epsilon_t)$ is decreasing in x_t, $\widehat{V}_t'(x_t)\le 0$ must hold. If $y_t^*=x_t$, then

$$\widehat{V}_t(x_t)=px_t+\mathbb{E}[\widehat{H}_t(x_t-d_t(x_t,\epsilon_t))],$$

which is concave in x_t. Then, $\widehat{V}_t'(x_t)\le\widehat{V}_t'(0)$. Note that $\widehat{V}_t(0)=\mathbb{E}[\beta\widehat{V}_{t+1}(0)]$. Since $\widehat{V}_{t+1}(x_{t+1})$ is decreasing in $x_{t+1}\ge 0$, $\widehat{V}_t'(0)=\beta\widehat{V}_{t+1}'(0)\le 0$. Then, in this case, $\widehat{V}_t'(x_t)\le 0$. As a result, we must have $\widehat{V}_t'(x_t)\le 0$ for any $x_t\ge 0$. ■

The Backorder Model with Quality-Dependent Demands

Let $\widehat{V}_t(\cdot)$ be the expected optimal discounted cost function from period t and onward. The dynamic recursion of this backorder model is

$$\widehat{V}_t(x_t)=\max_{0\le\theta_t\le 1,y_t\ge x_t}\widehat{U}_t(\theta_t,y_t),$$

where for $t=1,\cdots,T$

$$\begin{aligned}&\widehat{U}_t(\theta_t,y_t)\\ =&\int_{\underline{\epsilon}}^{\overline{\epsilon}}\Big[pd_t(\theta_t,\zeta)-K(\theta_t)+b(y_t-d_t(\theta_t,\zeta))-(b+h)[y_t-d_t(\theta_t,\zeta)]^++\beta\widehat{V}_t(y_t-d_t(\theta_t,\zeta))\Big]\,w_t(\zeta)d\zeta.\end{aligned}$$

The terminal condition is $\widehat{V}_{T+1}(x_{T+1})\equiv 0$. Again we adopt the following equivalent transformation for the dynamic recursion. Let $V_t(x_t)=\widehat{V}_t(x_t)+Mx_t$ and, for the ease of exposition, $G_t(y_t-d_t(\theta_t,\epsilon_t))=b(y_t-d_t(\theta_t,\epsilon_t))-(b+h)[y_t-d_t(\theta_t,\epsilon_t)]^++\beta V_{t+1}(y_t-d_t(\theta_t,\epsilon_t))$. Then,

$$V_t(x_t)=\max_{\theta_t\in[0,1],y_t\ge x_t}U_t(\theta_t,y_t)+Mx_t,$$

where

$$U_t(\theta_t,y_t)=\int_{\underline{\epsilon}}^{\overline{\epsilon}}\left[(p+\beta M)d_t(\theta_t,\zeta)-K(\theta_t)-\beta My_t+G_t(y_t-d_t(\theta_t,\zeta))\right]w_t(\zeta)d\zeta.$$

for $t=1,\cdots,T$. The terminal condition is $V_{T+1}(x_{T+1})\equiv Mx_{T+1}$.

Proof of Proposition 5.

Similar to the proof of Theorem 1, we can show the continuity of value functions in this model. We then use the inductive proof to show the results. Notice that $V_{T+1}(\cdot)$ is concave. Suppose that $V_{t+1}(\cdot)$ is concave, it suffices to show that $V_t(\cdot)$ is concave as well.

With $M=-b/\beta$, the joint concavity of $U_t(\theta_t,y_t)$ is guaranteed when (1) $\mathbb{E}[G_t(y_t-d_t(\theta_t,\epsilon_t))]$ is jointly concave in (θ_t,y_t), and (2) $\mathbb{E}[(p-b)d_t(\theta_t,\epsilon_t)]-K(\theta_t)$ is concave in θ_t.

Since $V_{t+1}(\cdot)$ is concave by the inductive assumption, $G_t(\cdot)$ is concave and

$$G_t'(x_{t+1})=\begin{cases}-h+\beta V_{t+1}'(x_{t+1}), & x_{t+1}\ge 0,\\ b+\beta V_{t+1}'(x_{t+1}), & x_{t+1}<0.\end{cases}$$

Based on the same argument in the proof of Lemma 2, we have $V_{t+1}'(x_{t+1})\le M$ and hence $G_t'(x_{t+1})\le 0$ for any x_{t+1}. Then, based on Proposition 1, $\mathbb{E}[G_t(y_t-d_t(\theta_t,\epsilon_t))]$ is jointly concave in (θ_t,y_t) in its domain. As $K(\theta_t)$ is increasingly convex in θ_t and $p-b>0$, $\mathbb{E}[(p-b)d_t(\theta_t,\epsilon_t)]-K(\theta_t)$ is concave in θ_t if

$$\mathbb{E}[(p-b)d_{t,\theta\theta}(\theta_t,\epsilon_t)]\le 0,$$

which must be true as $d_{t,\theta}(\theta_t, \epsilon_t)$ has the USDP/LSDP. As a result, $U_t(\theta_t, y_t)$ is jointly concave in (θ_t, y_t) and hence $V_t(x_t)$ is concave. Moreover, $U_t(\theta_t, y_t)$ is supermodular in (θ_t, y_t) as $d_t(\theta_t, \epsilon_t)$ is increasing in θ_t.

The continuity and concavity of the value functions imply that there exist constants $S_t = \arg\max_{y_t} U_t(\theta_t^*(y_t), y_t)$ and $\theta_t^*(y_t) = \arg\max_{\theta_t \in [0,1]} U_t(\theta_t, y_t)$ given y_t. Hence, a base stock list quality level policy is optimal, and the list quality level $\theta_t^*(y_t)$ is increasing in y_t. ■

The Lost-Sales Model with Quality-Dependent Demands

We formulate the dynamic recursion as follows:

$$\widehat{V}_t(x_t) = \max_{y_t \geq x_t} \left[p y_t - K(\theta_t) + \mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t, \epsilon_t))] \right]$$

for $t = 1, \cdots, T$, where

$$\widehat{H}_t(y_t - d_t(\theta_t, \zeta)) = -(p+h)[y_t - d_t(\theta_t, \zeta)]^+ + \beta \widehat{V}_{t+1}([y_t - d_t(\theta_t, \zeta)]^+),$$

and $\widehat{V}_{T+1}(x_{T+1}) \equiv 0$.

Proof of Proposition 6.

Without loss of generality, we assume that $d_t(\theta_t, \epsilon_t)$ is strictly decreasing in ϵ_t and $d_{t,\theta}(\theta_t, \epsilon_t)$ has the USDP. We can obtain the same results when $d_t(\theta_t, \epsilon_t)$ is strictly increasing in ϵ_t and $d_{t,\theta}(\theta_t, \epsilon_t)$ has the LSDP.

We still show the results by induction. Notice that $\widehat{V}_{T+1}(x_{T+1})$ must be decreasing and concave in $x_{T+1} \geq 0$. Suppose that $\widehat{V}_{t+1}(x_{t+1})$ is decreasing and concave in $x_{t+1} \geq 0$, it suffices to show that $\widehat{V}_t(x_t)$ is decreasing and concave in $x_t \geq 0$ as well.

As $\widehat{V}_{t+1}(\cdot)$ is decreasing and concave, $\widehat{H}_t(y_t - d_t(\theta_t, \zeta))$ must be decreasing and concave in $y_t - d_t(\theta_t, \zeta)$ and also in y_t. Notice that $K(\theta_t)$ is convex in θ_t. Based on Proposition 1, $\mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t, \zeta))]$, as well as $p y_t - K(\theta_t) + \mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t, \zeta))]$, is concave in y_t, θ_t, then $\widehat{V}_t(x_t)$ is concave.

Let

$$\theta_t^*(y_t) = \arg\max_{\theta_t \in [0,1]} \left[p y_t - K(\theta_t) + \mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t, \epsilon_t))] \right]$$

and

$$S_t = \arg\max \left[p y_t - K(\theta_t^*(y_t)) + \mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t^*(y_t), \epsilon_t))] \right].$$

Based on the joint concavity of the objective function, if $x_t < S_t$, then it is optimal to order up to S_t; otherwise, it is optimal to order nothing. As $d_t(\theta_t, \epsilon_t)$ is increasing in θ_t, then $\mathbb{E}[\widehat{H}_t(y_t - d_t(\theta_t, \epsilon_t))]$ is supermodular in (y_t, θ_t) as $\widehat{H}_t(\cdot)$ is concave. Then, $y_t^*(\theta_t)$ is increasing in θ_t according to Theorem 2.7.5 of Topkis (1998).

We then shall show that $\widehat{V}_t'(x_t) \leq 0$ for any $x_t \geq 0$. If $x_t < S_t$, then

$$\widehat{V}_t'(x_t) = 0.$$

Since V_t is concave, so we must have $\widehat{V}_t'(x_t) \leq 0$ for $x_t \geq S_t$. As a result, we must have $\widehat{V}_t'(x_t) \leq 0$ for any $x_t \geq 0$. ■

References

Boyd, S., L. Vandenberghe. 2004. Convex optimization. Cambridge University Press.

Federgruen, A., N. Yang. 2011. Procurement strategies with unreliable suppliers. *Operations Research*, **59**(4): 1033-1039.

Fiacco, A. V., J. Kyparisis. 1985. Sensitivity analysis in nonlinear programming under second order assumptions. A., Bagchi, H., Jongen (Eds). *Systems and Optimization*, **66**. Lecture Notes in Control and Information Sciences, Heidelberg, Springer, Berlin, 74-79.

Flanders, H. 1973. Differentiation under the integral sign. *The American Mathematical Monthly*, **80**(6): 615-627.

Milgrom, P., I. Segal. 2002. Envelope theorems for arbitrary choice sets. *Econometrica*, **70**(2): 583-601.

Steele, J. M. 2004. *The CauchySchwarz Master Class*, Cambridge University Press, United Kingdom.

Topkis, D. M. 1998. *Supermodularity and complementarity*, Princeton University Press.

Yeh, J. 2006. *Real Analysis: Theory of Measure and Integration*, World Scientific Publishing Company.

Appendix B: Additional Proof

Our results are still valid when $\underline{\epsilon} \to -\infty$ and $\overline{\epsilon} \to +\infty$ but require stronger conditions to ensure the twice continuous differentiability of $U_t(p_t, y_t)$ in both the lost-sales and the backorder models. We thus show the additional conditions in order to apply the dominated convergence theorem.

When the support of ϵ_t is $(-\infty, +\infty)$, the interchange of integration and differentiation operators is no longer straightforward. It is workable under certain conditions as shown in Federgruen and Yang (2011). In the following, we illustrate with the lost-sales model. The case under the backorder model can be similarly investigated but is much more simpler.

To ensure the interchange of integration and differentiation in the first order derivatives, i.e.,

$$\begin{aligned}\frac{\partial U_t(p_t,y_t)}{\partial p_t} =& y_t - \int_{\bar{\zeta}_t}^{+\infty} \left[y_t - d_t(p_t,\zeta) - (p_t+h)d_{t,p}(p_t,\zeta) + \beta \widehat{V}'_{t+1}(y_t - d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)\right] w_t(\zeta)d\zeta \\ \frac{\partial U_t(p_t,y_t)}{\partial y_t} =& p_t + \int_{\bar{\zeta}_t}^{+\infty} \left[-(p_t+h) + \beta \widehat{V}'_{t+1}(y_t - d_t(p_t,\zeta))\right] w_t(\zeta)d\zeta\end{aligned}$$

it requires that (1) $\widetilde{V}'_{t+1}(\cdot)$ exists, (2) the integrands are continuous functions of (p_t, y_t, ζ), (3) $\widehat{V}'_{t+1}(\cdot)$ is uniformly bounded, and (4) $-\infty < \int_{\bar{\zeta}_t}^{+\infty} d_{t,p}(p_t,\zeta)w_t(\zeta)d\zeta < +\infty$, $-\infty < \int_{\bar{\zeta}_t}^{+\infty} w_t(\zeta)d\zeta < +\infty$. Due to the assumptions we make, the conditions (1) and (2) are satisfied. Note that $|\widehat{V}'_{t+1}| \le \max\{(T-t)h, (T-t)\overline{p}\}$, i.e., it is uniformly bounded. Hence, the condition (3) is satisfied. For the condition (4), we require that $-\infty < d_{t,p}(p_t,\zeta) < +\infty$.

Let $Z_t(p_t,\zeta) = \frac{w_t(\zeta)}{d_{t,\zeta}(p_t,\zeta)}$. Then, through integrating by parts, we have

$$\begin{aligned}\frac{\partial U_t(p_t,y_t)}{\partial y_t} =& p_t - (p_t+h)\int_{\bar{\zeta}_t}^{+\infty} w_t(\zeta)d\zeta - \beta\left[Z_t(p_t,\zeta)\widehat{V}_{t+1}(y_t - d_t(p_t,\zeta))\right]_{\zeta\to+\infty} \\ &+ \beta\left[Z_t(p_t,\bar{\zeta}_t)\widehat{V}_{t+1}(y_t - d_t(p_t,\bar{\zeta}_t))\right] + \beta\int_{\bar{\zeta}_t}^{+\infty} \widehat{V}_{t+1}(y_t - d_t(p_t,\zeta))Z_{t,\zeta}(p_t,\zeta)d\zeta,\end{aligned}$$

where $Z_{t,\zeta}(p_t,\zeta)$ is the first order derivative of $Z_t(p_t,\zeta)$ w.r.t ζ. In the second order partial derivative $\partial^2 U_t(p_t,y_t)/\partial y_t \partial p_t$, if the interchange of integration operator and differentiation operator is permitted, the fifth item above becomes

$$\begin{aligned}&-\beta\int_{\bar{\zeta}_t}^{+\infty} \widehat{V}'_{t+1}(y_t - d_t(p_t,\zeta))d_{t,p}(p_t,\zeta)Z_{t,\zeta}(p_t,\zeta)d\zeta + \beta\int_{\bar{\zeta}_t}^{+\infty} \widehat{V}_{t+1}(y_t - d_t(p_t,\zeta))Z_{t,p\zeta}(p_t,\zeta)d\zeta \\ &-\beta\widehat{V}_{t+1}(y_t - d_t(p_t,\bar{\zeta}_t))Z_{t,\zeta}(p_t,\bar{\zeta}_t)\frac{\partial \bar{\zeta}_t}{\partial p_t}.\end{aligned}$$

Then, to make sure that the interchange is valid under the dominated convergence theorem, we require that (1) $-\infty < \int_{\bar{\zeta}_t}^{+\infty} d_{t,p}(p_t,\zeta)Z_{t,\zeta}(p_t,\zeta)d\zeta < +\infty$, (2) $d_t(p_t,\zeta)$ convergences to some limited value M_p for any feasible p_t when $\zeta \to +\infty$ so that $\widehat{V}_{t+1}(y_t - d_t(p_t,\zeta))$ is limited, and (3) $-\infty < \int_{\bar{\zeta}_t}^{+\infty} Z_{t,p\zeta}(p_t,\zeta)d\zeta < +\infty$. In addition, we require that $-\infty < \frac{w_t(\zeta)}{d^2_{t,\zeta}(p_t,\zeta)} < +\infty$ as $\zeta \to +\infty$ to ensure the continuity of the second order partial derivatives. Under these conditions, our results in the main body still hold.